Multistate analysis: tables of working life

F J Willekens

Mens en Ruimte and European Research Institute for Regional and Urban Planning, Froissartstraat 118, B-1040 Brussels, Belgium

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Abstract. The demographer's approach to the study of the labor force includes the calculation of life tables for the active population. Although this tool is extensively used and many countries publish working-life tables, and although they rely on very restrictive assumptions, the technique for constructing such tables has not improved since its development about thirty years ago. This paper reviews the conventional method for the construction of working-life tables and proposes a multi-state approach which does not rely on restrictive assumptions such as the unimodality of the labor-force participation curve. Instead of focusing on changes in stocks, the actual flows of people between active and inactive life are considered. The technique is compared with a similar procedure recently developed by Hoem and Fong. The increment-decrement table of working life serves as a basis for a multistate model for labor-force projection. The proposed model is compared with conventional approaches. The methods presented in this paper are illustrated using Danish data.

1 Introduction

At the St Louis meeting of the Population Association of America in 1977, Dr S J Smith of the US Bureau of Labor Statistics described an interesting example of the use of working-life tables, namely the estimation of lost earnings in the case of death before or during active life (Smith, 1977). This 'money-value of man' is of particular relevance in liability estimations. With the increase in the number of liability claims over the years, working-life tables now are used more than ever. Liability attorneys, and others interested in determining foregone earnings, rely on the concept of working-life expectancy as an input to their determinations.

On the other hand, working-life tables are used in macroeconomic analysis to describe the supply of labor and to study the impact of structural changes in the labor force, both in terms of changing activity rates and in terms of changing age compositions (Durand and Miller, 1968, page 19). Today working-life tables are being published for a large number of nations. An illustrative list is given by the United Nations (1973, page 318) and by Hoem and Fong (1976a, pages 6-7).

The expanding interest in the table of working life as a powerful analytical tool for studying the worklife potential of an individual or of a population forces us to view such tables in a new light and to rethink the way they are calculated. Smith concluded her paper at the St Louis meeting as follows:

"It is urged that the user of working life tables give careful consideration to the assumptions on which they rest, and the validity of those assumptions for his particular problem. These tables can be useful in many circumstances, but should never be thoughtlessly applied" (Smith, 1977, page 23).

The technique for constructing such a life table has not improved since the design of the first tables in the late 1940s (Wolfbein, 1949). Several authors, however, refer to the unrealistic assumptions on which this technique is based and to how they may affect the results. It is only recently that one observes methodological innovations. Hoem and Fong (1976a) have developed a working-life table which does not rely on the unrealistic assumptions required in conventional working-life tables.

It is a multistate increment–decrement life table which explicitly incorporates the gross flows of people between active and inactive life.

It is the purpose of this paper to reformulate the approach of Hoem and Fong in simpler terms and with less restrictive assumptions by applying recent findings of multiregional demography (Rogers, 1975). This requires some theoretical considerations on the construction of multistate life tables. First, however, I shall review the conventional method for calculating working-life tables.

2 The conventional table of working life

Besides the usual assumptions underlying a classical life table, conventional tables of working life generally adopt three additional general assumptions (Kpedekpo, 1974, page 292).

- (1) Persons who enter the labor force do so prior to the age at which the activity rate reaches its maximum. This implies that the labor-force participation (LFP) schedule has a maximum, that is, is unimodal.
- (2) Prior to the age of maximum labor-force participation, no survivors retire from the labor force and become members of the inactive population. Retirement only occurs at ages beyond the age of the maximum activity rate. Once a person has left the labor force, he can never return.
- (3) The rates of mortality at each age are the same for economically active and inactive persons.

The first two assumptions are usually satisfied for males but not for females. The male LFP curve has a very regular pattern: labor-force participation starts at about the age of fifteen and reaches a peak around the age of thirty. Between the mid-thirties and the mid-forties the proportion of the male population in the labor force drops gradually, and then declines rapidly due to retirement. There is much less regularity in the female LFP schedules. They are usually bimodal, particularly in North America and in European countries. This is due to the patterns of entry into the labor force and of withdrawal or retirement, which are related to the life cycle of marriage and fertility. Women may drop out of the labor force to marry and have children and enter it again later when the children no longer need their close attention. Divorce and widowhood are also events determining the age of (re)entry into the labor force [for a discussion of LFP curves by demographic and social categories, and of their patterns of change, see Bowen and Finegan (1969) and Durand (1975)]. The problem of bimodality of female LFP curves may be eliminated by constructing working-life tables for women by marital status (Garfinkle, 1967; Smith, 1977, page 9). Such tables are regularly calculated by the US Bureau of Labor Statistics.

The third assumption is generally not true, since the age-specific mortality of an active population generally exceeds that of an inactive population. All three assumptions may be dropped by constructing multistate tables of working life, which will be the task of the next section.

A typical working-life table represents the life history of a hypothetical population or cohort. In addition to the mortality experience, it describes the pattern of labor-force participation. All the columns are derived by applying the mortality rates and LFP rates to a hypothetical population or cohort. The LFP rates may be replaced by rates of labor-force entry and of separation by use of assumptions (1) and (2) (Fullerton, 1971, pages 51–52). The computational procedure of constructing a working-life table has been given by Wolfbein (1949), by Durand and Miller (1968, annex A), and by Fullerton (1971, pages 52–54), among others. The procedure is summarized in the remainder of this section.

2.1 Number of persons living, $L(x)$

Let $L(x)$ denote the number of persons aged x to $x+1$ in the stationary life-table population. They are computed from the age-specific mortality rates only and appear in all standard life tables. Usually $L(x)$ is expressed per 100 000 births (that is, the radix is 100 000). Besides the number-of-persons interpretation of $L(x)$, $L(x)$ also may be interpreted as the number of years expected to be lived between ages x and $x+1$ by the cohort. The two interpretations, number of people and person-years, are used interchangeably in demography.

2.2 Number of persons in the labor force, $L^w(x)$

The age composition of the labor force in the stationary population is derived as the product

$$L^w(x) = w(x)L(x) , \quad (1)$$

where $w(x)$ is the age-specific activity rate. Note that $L^w(x)$ also represents the expected time spent in active life between ages x and $x+1$ by the cohort. It should be remarked that, although most authors derive the working-life table from the $L(x)$ column of the life table, some use the $l(x)$ column, the number of people at *exact* ages x (Fullerton, 1971; 1972; Kpedekpo, 1969).

Summing the expected time spent in active life between all pairs of ages beyond age x gives

$$T^w(x) = \sum_{y=x}^z L^w(y) , \quad (2)$$

where z denotes the final age interval. It is the total expected time spent in active life beyond age x by the cohort of 100 000 people. The variable $T^w(x)$ is analogous to the total person-years lived beyond age x in the conventional life table. For ages below $\bar{\alpha}$, the minimum age for working activity, $T^w(x)$ is identical and equal to $T^w(0)$. The index $T^w(\bar{\alpha})$, when expressed per unit cohort, has been labeled the *net years of active life* (Farooq, 1975, page 45). An index of the time spent in active life in the absence of mortality is the *gross years of active life* (GYA) index (Farooq, 1975, page 44; Durand, 1975, page 226) and is given by $\sum_x w(x)$. The area under the LFP curve gives the GYA and this area is analogous to the gross reproduction rate in fertility analysis. The difference between GYA and $T^w(\bar{\alpha})$ represents the loss in working life due to mortality. It has been estimated for males as 4.8 years in industrialized countries, 8.5 years in semi-industrialized countries, and 11.4 years in agricultural countries (United Nations, 1973, page 319).

2.3 Expectation of working life, $e^w(x)$

The average remaining number of years of working life or the expectation of working life beyond age x is

$$e^w(x) = \left[\sum_{y=x}^z L^w(y) \right] [l^w(x)]^{-1} . \quad (3)$$

It defines the average number of years of working life remaining to a person *in the labor force* at exact age x . The value of $l^w(x)$ is computed as follows (Wolfbein, 1949, page 291; Fullerton, 1971, page 54): if $w(k)$ is the maximum labor-force participation rate, attained at age k , then the value of $l^w(x)$ is given by

$$l^w(x) = \frac{1}{2} [L^w(x-1) + L^w(x)] , \quad \text{for } x > k , \quad (4)$$

and by

$$l^w(x) = \frac{1}{2} [L(x-1) + L(x)] w(k) , \quad \text{for } x \leq k . \quad (5)$$

The consideration of the maximum activity rate for ages below or equal to k is made to eliminate the effect of entries into the labor force in the years following age x . The working-life expectancy at age x refers to the cohort of active population $l^w(x)$. Therefore additional entries into the labor force after x may not be considered. The implicit assumption is that all work is done by a distinct cohort of workers. The application of $w(k)$ for $x \leq k$ reflects the assumption that all entries into the labor force occur at the youngest labor-force age, $\bar{\alpha}$ (sixteen years say). Between ages $\bar{\alpha}$ and k , no person is supposed to leave the labor force [assumption (2)]. Hence the active population of exact age k is smaller than at age $\bar{\alpha}$ only because of mortality in the intervening years.

Several authors use two measures of working life in their analysis (for example, Wolfbein, 1949, page 293; Fullerton and Byrne, 1976, page 35; Farooq, 1975, page 46). The first, the so-called *labor-force-based measure*, is identical to $e^w(x)$ and is sometimes called the *average remaining number of years of active life*. The second, the *population-based measure*, is the ratio $\left[\sum_{y=x}^z L^w(y) \right] [l(x)]^{-1}$, where $l(x)$ is the total number of people of exact age x in the life table, and is known as the *expectation of active life*. The second measure assumes that all persons in the population, currently active or inactive, have an equal probability of participating in the labor force. The two measures serve different purposes. If one is interested in the working-life expectancy of a person not yet in the labor force, or of a person regardless of his current labor-force status, the population-based measure is appropriate. This approach has been used by Durand (1948, pages 259–265) in estimating what he calls “the average number of years in the labor force”. However, if one is interested in the remaining years of work of a currently active person, the labor-force-based measure will be more accurate (for example, see Durand and Miller, 1968, pages 24–27). For example, the expected length of working life at birth is

$$e^w(0) = \left[\sum_{y=0}^z L^w(y) \right] [l(0)]^{-1}. \quad (6)$$

2.4 Rate of accession to the labor force, $A(x)$

This measure shows the net accessions to the life-table labor force between ages x to $x+1$ as a ratio of the life-table population, $L(x)$. It gives the proportion of the population aged x to $x+1$ in a life-table cohort who are not currently in the labor force but who will engage in labor activity in the next year. The ratio of the net accessions to the life-table labor force is (Wolfbein, 1949, page 288)

$$A(x) = \frac{L^w(x+1) - L^w(x)[1 - M(x)]}{L(x)}, \quad (7)$$

where $M(x)$ is the age-specific mortality rate, and the product $L^w(x)M(x)$ is the mortality in the labor force between ages x and $x+1$. The quantity $A(x)$ is not computed for ages above k , since it has been assumed that people enter the labor force only up to age k , at which age the LFP rate is at its maximum.

2.5 Rate of separation from the labor force, $M^w(x)$

The rate of separation due to all causes (mortality and retirement) is defined as the ratio of the net separation from the labor force between ages x and $x+1$ to the stationary labor force, $L^w(x)$:

$$M^w(x) = \frac{l^w(x) - l^w(x+1)}{L^w(x)}. \quad (8)$$

This measure is very similar to the death rate in a conventional life table. Before age k it is assumed that withdrawal from the labor force is due to mortality only. After age k , two types of separation occur: mortality and retirement. The rate of separation due to mortality is

$$M_{\delta}^w(x) = \frac{l^w(x)q(x)}{L^w(x)}, \quad (9)$$

where $q(x)$ is the probability of dying between ages x and $x+1$ (identical for an active and an inactive population). The rate of separation due to retirement is a residual:

$$M_r^w(x) = M^w(x) - M_{\delta}^w(x). \quad (10)$$

Separation rates are important for manpower planning because they permit the calculation of expected losses from active life due to death and retirement (see, for example, Garfinkle, 1967).

2.6 Replacement

The difference between the total accession rate and the total separation rate is the *labor-force replacement rate* (United Nations, 1973, page 319; Farooq, 1975, page 52). The ratio of accessions to separations is the *replacement ratio*, a measure of pressure on the labor market. A ratio of less than one shows that not all vacancies caused by death and retirement are being filled.

3 The increment-decrement table of working life: methodology

The conventional method for constructing tables of working life is based on a number of unrealistic assumptions, such as the unimodality of the LFP curve. Data availability and existent methodology may have required these assumptions. The method presented here introduces more realism into the working-life table, but at the same time it increases the data requirements. Today most countries do not publish all the necessary data; therefore, parallel to the development of the life-table methodology, attention should be devoted to procedures for estimating missing data.

The working-life table proposed in this study is an increment-decrement life table. Recently there has been renewed interest in increment-decrement life tables (for example, Schoen, 1975) and in their formulation by the application of the theory of multiregional mathematical demography (Rogers, 1973; 1975; Rogers and Ledent, 1976; Ledent, 1978; 1980). Also in this paper, multiregional demographic techniques are used for deriving increment-decrement tables of working life. The method presented has a number of conceptual similarities with a technique developed recently by Hoem and Fong (1976a); however, the mathematics are somewhat simpler and regional differences in mortality can be easily handled.

One begins with a population, disaggregated by age and sex, that is partitioned into two groups: active population (labor force) and inactive population. The analysis here will be performed for a single sex only, although the extension to a two-sex model is straightforward. These are the three states of the system. Everyone starts out as a member of the inactive population, state 1. Labor-force membership is denoted by state 2. State 3, denoted by δ , is the state of death. In each age group, people may move between state 1 and state 2 (transient states), and into state δ (absorbing state). The age-specific *gross flows* in and out of the labor force are explicitly taken into account. The focus on gross flows represents a fundamental difference from the conventional technique of constructing working-life tables, with its emphasis on the size (stock) of the labor force and on changes in the stock through *net* flows. The assumption, made in conventional working-life tables, that

everyone enters the labor force before age k and retires at ages over k , is therefore no longer necessary. Consequently the need for a unimodal LFP curve no longer exists.

The increment–decrement table of working life represents the mortality and mobility experience of a cohort. The mobility experience represents movement into and out of the labor force. All life-table statistics are derived from a set of age-specific mobility and mortality probabilities.

The usual procedure of life-table construction is followed, beginning with the expression of probabilities in terms of instantaneous rates of mobility and mortality (also known as intensities or forces of mobility and mortality). Let $\mu_{12}(x)$ denote the instantaneous rate of accession to the labor force at age x : the rate at which people of age x enter the labor force. The instantaneous rate is the limiting value of the accession rate as the age interval becomes infinitesimal. The instantaneous rate of separation from the labor force at age x is $\mu_{21}(x)$, and the forces of mortality are given by $\mu_{i\delta}(x)$, where $i = 1, 2$. The force of mortality of the active population may be different from that of the inactive population. The assumption of equal mortality rates, which is implicit in conventional tables of working life, is therefore no longer needed.

3.1 The transition probabilities

The transition from one state to another is defined by the Kolmogorov equation. Consider the cohort of people who are now of exact age y . Denote the probability that an individual of age y in state 1 will be in state 2 n years later by ${}_y l_2(y+n)$. Denoting $y+n$ by x ($x \geq y$), this probability may be written as ${}_y l_2(x)$. The probability that a person in state 1 at age y will be in state 2 at age $x+dx$ is

$${}_y l_2(x+dx) = {}_y l_1(x)\mu_{12}(x)dx + {}_y l_2(x)\mu_{22}(x)dx \tag{11}$$

where $\mu_{12}(x)dx$ is the probability that an x -year-old member of the inactive population will become active within time dx and $\mu_{22}(x)dx$ is the probability that an x -year-old member of the labor force will still be active dx years (or time units) later. It is assumed that only one interstate ‘passage’ is possible in the small interval dx .

The transition probabilities are independent of the status of the individual at age y , but do depend on his status at exact age x , that is, at the beginning of the interval $(x, x+dx)$. This is the Markovian assumption. Let

$$\mu_{22}(x)dx = 1 - [\mu_{2\delta}(x) + \mu_{21}(x)]dx \tag{12}$$

Then equation (11) may be written as

$${}_y l_2(x+dx) = {}_y l_1(x)\mu_{12}(x)dx + {}_y l_2(x) - {}_y l_2(x)[\mu_{2\delta}(x) + \mu_{21}(x)]dx \tag{13}$$

or as

$$\frac{{}_y l_2(x+dx) - {}_y l_2(x)}{dx} = {}_y l_1(x)\mu_{12}(x) - {}_y l_2(x)[\mu_{2\delta}(x) + \mu_{21}(x)] \tag{14}$$

which is the Kolmogorov differential equation

$$\frac{d}{{dx}} {}_y l_2(x) = {}_y l_1(x)\mu_{12}(x) - {}_y l_2(x)[\mu_{2\delta}(x) + \mu_{21}(x)] \tag{15}$$

Equation (15), which may also be found in Hoem and Fong (1976a, page 69), describes the changes in ${}_y l_2(x)$ as a function of the instantaneous rates of mobility and mortality and of the initial condition, namely the state of a person at exact age x .

Paralleling the derivation of equation (15), one may obtain expressions for changes in ${}_y l_1(x)$, ${}_2 y l_1(x)$, and ${}_2 y l_2(x)$. The result may be expressed in matrix form:

$$\frac{d}{{dx}} {}_y l(x) = -\mu(x){}_y l(x) \tag{16}$$

where $\mu(x)$ is the matrix of instantaneous rates,

$$\mu(x) = \begin{bmatrix} \mu_{1\delta}(x) + \mu_{12}(x) & -\mu_{21}(x) \\ -\mu_{12}(x) & \mu_{2\delta}(x) + \mu_{21}(x) \end{bmatrix}, \quad (17)$$

and

$${}_y\mathbf{l}(x) = \begin{bmatrix} {}_{1y}l_1(x) & {}_{2y}l_1(x) \\ {}_{1y}l_2(x) & {}_{2y}l_2(x) \end{bmatrix}, \quad (18)$$

where ${}_{iy}l_j(x)$ is the probability that a person aged y in state i will be in state j n years later, that is, at exact age x ($n = x - y$). The age y denotes the cohort. If $y = 0$ the cohort considered is the birth cohort or radix. Another interesting cohort is $y = \bar{\alpha}$, the lowest age of labor-force participation.

The solution to equation (16) is of the form

$${}_y\mathbf{l}(x) = {}_y\mathbf{\Omega}(x){}_y\mathbf{l}(y), \quad (19)$$

where ${}_y\mathbf{\Omega}(x)$ is the transition matrix, which was known in early literature as the *matricant* (Gantmacher, 1959).

The determination of the transition matrix of a linear time-variant system of differential equations, such as system (19) has received considerable attention in systems-theory literature, from which the necessary analytic results can be drawn (for example, Wolovich, 1974). The principal problem is to relate the transition matrix to the instantaneous rates. One may generalize the approach adopted in single-region life-table construction and try the following 'solution' to equation (16):

$${}_y\mathbf{l}(x) = \exp\left[\int_y^x -\mu(t)dt\right]{}_y\mathbf{l}(y). \quad (20)$$

However, equation (20) is a solution to equation (16) only if $\mu(t)$ commutes for all t , a restriction which in general does not hold. Two approaches will be considered here in order to get around this problem.

The first approach is to introduce the assumption that $\mu(t)$ is constant in the interval $x-h$ to x , that is, that $\mu(t) = \mu(x-h)$ for $x-h \leq t < x$ (Hoem and Fong, 1976a; Krishnamoorthy, 1979; Ledent, 1980). This implies that the continuous matrix function $\mu(t)$ is approximated by a step function. The heights of these steps between ages $x-h$ and x are given by the elements of $\mu(x-h)$. Under this assumption equation (20) becomes

$${}_y\mathbf{l}(x) = \exp[-h\mu(x-h)]{}_y\mathbf{l}(x-h). \quad (21)$$

The problem of solving equation (16) is now reduced to evaluating the exponential of the matrix $-h\mu(x-h)$. The evaluation of a function of the form $\exp(\mathbf{A})$ is a major issue in mathematical systems theory, and a number of procedures have been developed (Wolovich, 1974). One is to use the expansion

$$\exp(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \frac{1}{2!} \mathbf{A}^2 + \frac{1}{3!} \mathbf{A}^3 + \dots, \quad (22)$$

which applied to equation (21) gives

$${}_y\mathbf{l}(x) = \left\{ \mathbf{I} - h\mu(x-h) + \frac{1}{2!} h^2 [\mu(x-h)]^2 - \dots \right\} {}_y\mathbf{l}(x-h). \quad (23)$$

This procedure is adopted by Krishnamoorthy (1979) and Ledent (1980).

A second approach to solving equation (16), however, is followed in this paper. The following theorem is used. It is stated here without proof, but a proof can be found in Brauer et al (1970, pages 312–313). A simplified solution to equations (15)

and (16) may be derived if one assumes no mortality or equal mortality in both states (see the appendix).

Theorem: Solving the system of differential equations

$$\frac{d}{dt} \mathbf{Y}(t) = \mathbf{A}(t) \mathbf{Y}(t) , \quad \mathbf{Y}(t_0) \text{ given}, \quad (24)$$

is equivalent to finding a continuous matrix function $\mathbf{Z}(t)$ such that

$$\mathbf{Z}(t) = \mathbf{Y}(t_0) + \int_{t_0}^t \mathbf{A}(\tau) \mathbf{Z}(\tau) d\tau . \quad (25)$$

Hence equation (16) may be replaced by the integral equation

$${}_y l(x) = {}_y l(y) - \int_0^n \mu(y+t) {}_y l(y+t) dt , \quad (26)$$

where $n = x - y$. The problem of solving the system of Kolmogorov differential equations (16) has now been replaced by the problem of evaluating the integral $\int_0^n \mu(y+t) {}_y l(y+t) dt$. This may be done in steps of length h , say:

$${}_y l(y+h) = {}_y l(y) - \int_0^h \mu(y+t) {}_y l(y+t) dt , \quad (27)$$

$${}_y l(y+2h) = {}_y l(y+h) - \int_0^h \mu(y+h+t) {}_y l(y+h+t) dt , \quad (28)$$

$\vdots$

$${}_y l(x) = {}_y l(x-h) - \int_0^h \mu(x-h+t) {}_y l(x-h+t) dt , \quad (29)$$

$${}_y l(x+h) = {}_y l(x) - \int_0^h \mu(x+t) {}_y l(x+t) dt . \quad (30)$$

Define the matrix of conditional probabilities

$$\mathbf{P}(x) = \begin{bmatrix} p_{11}(x) & p_{21}(x) \\ p_{12}(x) & p_{22}(x) \end{bmatrix} , \quad (31)$$

where an element $p_{ij}(x)$ denotes the probability that a person alive in state i at age x will be in state j h years later, that is, at age $x+h$. The probabilities $p_{ij}(x)$ are assumed to be independent of the state at age y and only depend on the state at the beginning of the interval h . $\mathbf{P}(x)$ may also be written as ${}_x \Omega(x+h)$. The probability of dying is obtained by subtraction:

$$q_i(x) = 1 - p_{i1}(x) - p_{i2}(x) . \quad (32)$$

For a nonsingular ${}_y l(x)$, $\mathbf{P}(x)$ is given by the following expression:

$$\mathbf{P}(x) = {}_y l(x+h) [{}_y l(x)]^{-1} . \quad (33)$$

The nonsingularity condition is generally satisfied for ages $x \geq \bar{\alpha}$. Since no persons are in the labor force at the younger ages, whenever $x < \bar{\alpha}$ we have $p_{21}(x) = p_{12}(x) = p_{22}(x) = 0$ and $p_{11}(x)$ is simply the probability of surviving from age x to $x+h$. This probability is equal to

$$p_{11}(x) = \frac{{}_y l_1(x+h)}{{}_y l_1(x)} , \quad \text{for } x < \bar{\alpha} . \quad (34)$$

In other words matrix expression (33) reduces to a scalar expression when $x < \bar{\alpha}$.

Equation (33) states that ${}_y l(x+h) = \mathbf{P}(x) {}_y l(x)$. A comparison of this expression with equation (30) yields the following for $\mathbf{P}(x)$:

$$\mathbf{P}(x) = \mathbf{I} - \int_0^h \boldsymbol{\mu}(x+t) {}_y l(x+t) [{}_y l(x)]^{-1} dt. \quad (35)$$

Note the difference between equations (35) and (21), in which $\mathbf{P}(x) = \exp(-h \boldsymbol{\mu}(x-h))$. Which of the two estimations yields the best result is difficult to say without some additional information about the underlying mortality and mobility schedules.

From equation (30) one may also derive an expression for the annual age-specific rates in the increment-decrement life table. Note that

$${}_y l(x+h) - {}_y l(x) = - \left[\int_0^h \boldsymbol{\mu}(x+t) {}_y l(x+t) dt \right] \left[\int_0^h {}_y l(x+t) dt \right]^{-1} \left[\int_0^h {}_y l(x+t) dt \right]. \quad (36)$$

The expression $\int_0^h {}_y l(x+t) dt$ denotes the number of years lived in each state between ages x and $x+h$ per person in each state at age y . Denote this by ${}_y \mathbf{L}(x)$,

$${}_y \mathbf{L}(x) = \int_0^h {}_y l(x+t) dt. \quad (37)$$

The matrix ${}_y \mathbf{L}(x)$ may also be looked upon as representing the number of people by state in age group x to $x+h$ per person in each state at age y . Hence it gives the age distribution of the stationary life-table population. The expression

$$\left[\int_0^h \boldsymbol{\mu}(x+t) {}_y l(x+t) dt \right] \left[\int_0^h {}_y l(x+t) dt \right]^{-1} = \mathbf{m}(x) \quad (38)$$

is then the matrix of age-specific life-table rates [note the difference in the expression obtained by Ledent (1980), which is derived not from the Kolmogorov equation but from his particular definition of the elements of $\mathbf{m}(x)$]. Equation (36) becomes therefore

$${}_y l(x+h) - {}_y l(x) = -\mathbf{m}(x) {}_y \mathbf{L}(x). \quad (39)$$

To derive an expression for $\mathbf{P}(x)$ in terms of life-table rates, it should be recalled that ${}_y l(x+h) = \mathbf{P}(x) {}_y l(x)$. Hence

$$[\mathbf{P}(x) - \mathbf{I}] {}_y l(x) = -\mathbf{m}(x) {}_y \mathbf{L}(x), \quad (40)$$

and so

$$\mathbf{P}(x) = \mathbf{I} - \mathbf{m}(x) {}_y \mathbf{L}(x) [{}_y l(x)]^{-1}. \quad (41)$$

This expression may also be derived directly from equation (36) by applying equations (37) and (38).

3.2 Numerical approximation of $\mathbf{P}(x)$

All life-table statistics may be derived from the transition probability matrices $\mathbf{P}(x)$. The expression of $\mathbf{P}(x)$ in terms of instantaneous rates is given in equations (35) and (41). On the other hand, Rogers and Ledent (1976) have shown that $\mathbf{P}(x)$ may be written as follows:

$$\mathbf{P}(x) = \left[\mathbf{I} + \frac{h}{2} \mathbf{M}(x) \right]^{-1} \left[\mathbf{I} - \frac{h}{2} \mathbf{M}(x) \right], \quad (42)$$

where $\mathbf{M}(x)$ is the matrix of observed (empirical) rates, associated with age group x to $x+h$, set out in the same format as the $\boldsymbol{\mu}(x)$ matrix.

The transition from equation (41) to equation (42) involves a number of assumptions. The Markovian assumption, introduced earlier, states that the transition probability during a certain interval depends only on the state in which the person is at the

beginning of the interval and is independent of previous states. Another assumption is that transitions and deaths are distributed uniformly over the interval x to $x+h$. The mean duration of transfers is therefore $\frac{1}{2}h$. The uniform distribution implies that the integral $\int_0^h {}_y l(x+t) dt = {}_y L(x)$ may be approximated by linear integration (*the linear integration hypothesis*):

$${}_y L(x) = \frac{h}{2} [{}_y l(x) + {}_y l(x+h)] . \quad (43)$$

Introducing equation (43) in the expression for $P(x)$ yields

$$P(x) = I - \frac{h}{2} m(x) [{}_y l(x) + {}_y l(x+h)] [{}_y l(x)]^{-1} = I - \frac{h}{2} m(x) [I + P(x)] , \quad (44)$$

whence

$$P(x) + \frac{h}{2} m(x) P(x) = I - \frac{h}{2} m(x) \quad (45)$$

and

$$P(x) = \left[I + \frac{h}{2} m(x) \right]^{-1} \left[I - \frac{h}{2} m(x) \right] . \quad (46)$$

To compute $P(x)$ from observed data, a third assumption is introduced: that annual age-specific life-table rates are equal to the annual age-specific rates of the observed population, that is, $m(x) = M(x)$. This assumption implies that cross-sectional data (observed rates) provide adequate estimates of the life-table rates to be used in longitudinal analysis. It makes equation (46) equal to equation (42). According to Ledent (1980), equation (46) only holds for the movement approach. In the numerical illustration in section 4, the time interval of one year is short enough to assume that everyone makes only one move during the interval, and hence the transition approach coincides with the movement approach.

3.3 Other life-table statistics

Several useful statistics of the working-life table may be derived from the matrices of probabilities ${}_y l(x)$ and $P(x)$. The procedures are analogous to the construction of a multiregional life table in which states refer to regions (Rogers, 1975, chapter 3). The matrix ${}_y L(x)$ has already been derived. For example, the average number of years spent in the labor force between ages x and $x+h$ by an active person of exact age y is

$${}_y L_2(x) = \int_0^h {}_y l_2(x+t) dt , \quad (47)$$

which may be approximated by the linear interpolation

$$\frac{h}{2} [{}_y l_2(x) + {}_y l_2(x+h)] . \quad (48)$$

The total number of years expected to be spent in the labor force beyond age y by a person of age y already in the labor force is

$${}_y e_2(y) = \int_0^{\bar{\beta}-y} {}_y l_2(y+t) dt , \quad (49)$$

where $\bar{\beta}$ is the highest age of active life (about seventy years). The working-life expectancy of a person not active at age y is

$${}_1 e_2(y) = \int_0^{\bar{\beta}-y} {}_1 l_2(y+t) dt . \quad (50)$$

In an analogous way one may formulate the expected time spent outside the labor force before age $\bar{\beta}$ by a person who is active (inactive) at age y . The results may be set out as the matrix expression

$${}_y\mathbf{e}(y) = \int_0^{\bar{\beta}-y} {}_y\mathbf{l}(y+t) dt, \quad (51)$$

where

$${}_y\mathbf{e}(y) = \begin{bmatrix} {}_{1y}e_1(y) & {}_{2y}e_1(y) \\ {}_{1y}e_2(y) & {}_{2y}e_2(y) \end{bmatrix}. \quad (52)$$

The column sum ${}_{1y}e_*(y)$ denotes the total life expectancy of a person outside the labor force at age y . A part of this person's remaining lifetime will be spent in inactivity [${}_{1y}e_1(y)$]. The column sum ${}_{2y}e_*(y)$ refers to the life expectancy of a person in the labor force at age y . He will spend ${}_{2y}e_1(y)$ and ${}_{2y}e_2(y)$ years in active and inactive life respectively. The measure of ${}_{2y}e_2(x)$ is analogous to the labor-force-based measure of active life developed in the conventional tables of working life.

The population-based measure of working life is defined as follows:

$${}_ye_2(y) = \int_0^{\bar{\beta}-y} [{}_{1y}l_2(y+t) + {}_{2y}l_2(y+t)] dt. \quad (53)$$

It is clear that, for $y < \bar{\alpha}$, ${}_ye_2(y) = {}_{1y}e_2(y)$.

Formulas (51) and (53) refer to the life expectancy beyond age y of people aged y years. What is the life expectancy beyond age x of people in a given state at age y ? Since the state at age x is of no importance, one first computes the distribution of the x -year-old people by their state n years ago, that is, at age y . This is a vector ${}_y\mathbf{l}_*(x)$ computed as follows:

$${}_y\mathbf{l}_*(x) = [{}_y\mathbf{l}(x)]^T \mathbf{I}, \quad (54)$$

where T denotes transpose and $\mathbf{I}$ is a vector of ones. The life expectancy beyond age x by future state and by state at age y is

$${}_y\hat{\mathbf{e}}(x) = \left[\int_0^{\bar{\beta}-x} {}_y\mathbf{l}(x+t) dt \right] [{}_y\hat{\mathbf{l}}(x)]^{-1}, \quad (55)$$

where ${}_y\hat{\mathbf{l}}(x)$ is a diagonal matrix with the elements of ${}_y\mathbf{l}_*(x)$ along the diagonal. It is analogous to the matrix formula of life expectancy by place of birth in the multi-regional life table (Willekens, 1977b, page 656). Instead of regions and birth cohorts, we are considering here states and cohorts of people aged y at a given point in time. This shows how the state-specific life expectancy of a given cohort changes as age increases.

Knowing the values ${}_y\mathbf{l}(y+t)$, $t \geq 0$, for a given cohort, one may derive the life expectancy of a person in a given state at age x (and not y). This measure is analogous to the expectation of life by place of residence in multiregional demography. In working-life tables, the place of residence at age $y+n$ is replaced by the state at age x . The life expectancy by state at age x is

$${}_x\mathbf{e}(x) = \left[\int_0^{\bar{\beta}-x} {}_y\mathbf{l}(x+t) dt \right] [{}_y\mathbf{l}(x)]^{-1}. \quad (56)$$

This expression enables one to derive an interesting recursive expression for the expectation of life (Hoem and Fong, 1976a). One may rewrite equation (56) as

follows:

$$\begin{aligned} {}_xe(x) &= \left[\int_0^h {}_yl(x+t)dt + \int_0^{\beta-x-h} {}_yl(x+h+t)dt \right] [{}_yl(x)]^{-1} \\ &= {}_xL(x) + \left[\int_0^{\beta-x-h} {}_yl(x+h+t)dt \right] [{}_yl(x+h)]^{-1} [{}_yl(x+h)][{}_yl(x)]^{-1}, \end{aligned} \quad (57)$$

or equivalently

$${}_xe(x) = {}_xL(x) + {}_{x+h}e(x+h)P(x). \quad (58)$$

For the last age group, z ,

$${}_ze(z) = {}_zL(z) = [M(z)]^{-1}. \quad (59)$$

4 The increment-decrement table of working life: numerical illustrations

The increment-decrement table of working life may be derived from a set of age-specific accession rates, separation rates, and mortality rates. The required data for such a life table have been provided by Hoem and Fong (1976b, pages 7 and 12) and are based on Danish labor-force panel surveys during the period 1972–1974. Although sex-specific data are provided, the analysis is performed for the male population only. The data are repeated in the first three columns of table 1. They are not the observed rates but are computed by a step function and smoothed by a moving-average technique (Hoem and Fong, 1976a, pages 44–48). No mortality differentials have been assumed between the active and the inactive population. Although this assumption is not required, it has been made to allow comparison of these results with those published by Hoem and Fong (1976b).

Transition probabilities are shown in the last four columns of table 1⁽¹⁾. They are computed using equation (42). The transition probabilities associated with age group x depend only on the mobility and mortality rates of the same age group. For example, the probability matrix $P(x)$ for age group 20 to 21 is

$$P(20) = [I + \frac{1}{2}M(20)]^{-1}[I - \frac{1}{2}M(20)],$$

where

$$M(20) = \begin{bmatrix} 0.001221 + 0.457690 & -0.092260 \\ -0.457690 & 0.001221 + 0.092260 \end{bmatrix}.$$

Hence

$$P(20) = \begin{bmatrix} 0.6402 & 0.0723 \\ 0.3586 & 0.9265 \end{bmatrix}.$$

The probability that a person aged 20 will enter the labor force in the next year is 0.3586; the probability that he will survive but remain inactive is 0.6402.

For the present calculations it is important to note two points. First, the probability of separating from the labor force for 16 year olds is not zero. This is caused by the fact that people enter the labor force and drop out in the same year. The drop-out rate of young people is relatively high (figure 1). Second, for some age groups the transition probabilities $p_{12}(x)$ and $p_{21}(x)$ differ considerably from those obtained by Hoem and Fong. However, the probabilities of dying, obtained as residuals, are similar.

⁽¹⁾ The computer program used for the calculation of the increment-decrement table of working life is a slightly modified version of the multiregional life-table program listed in Willekens and Rogers (1978).

Table 1. Age-specific rates of mortality, accession, and separation, and transition probabilities.

Age	Age-specific rate			Transition probability			
	mortality	1 to 2	2 to 1	1 to 1	1 to 2	2 to 1	2 to 2
16	0.000733	0.339080	0.525690	0.7627	0.2366	0.3668	0.6325
17	0.000934	0.605350	0.171700	0.5635	0.4356	0.1236	0.8755
18	0.001202	0.635730	0.135850	0.5405	0.4583	0.0979	0.9009
19	0.001327	0.557140	0.121990	0.5832	0.4154	0.0910	0.9077
20	0.001221	0.457690	0.092260	0.6402	0.3586	0.0723	0.9265
21	0.001016	0.373170	0.085200	0.6957	0.3033	0.0693	0.9297
22	0.000944	0.345440	0.080600	0.7145	0.2845	0.0664	0.9327
23	0.000997	0.341970	0.071500	0.7159	0.2831	0.0592	0.9398
24	0.001014	0.355140	0.057370	0.7048	0.2941	0.0475	0.9515
25	0.000932	0.393230	0.041280	0.6763	0.3228	0.0339	0.9652
26	0.000901	0.423520	0.032250	0.6545	0.3446	0.0262	0.9729
27	0.000980	0.449790	0.024180	0.6357	0.3633	0.0195	0.9795
28	0.001033	0.474040	0.019460	0.6191	0.3799	0.0156	0.9834
29	0.001039	0.492260	0.015090	0.6067	0.3923	0.0120	0.9870
30	0.001066	0.499390	0.012050	0.6016	0.3973	0.0096	0.9893
31	0.001198	0.495120	0.012120	0.6043	0.3945	0.0097	0.9891
32	0.001319	0.478300	0.012530	0.6151	0.3836	0.0100	0.9886
33	0.001284	0.451480	0.012780	0.6327	0.3660	0.0104	0.9884
34	0.001260	0.416880	0.010740	0.6557	0.3431	0.0088	0.9899
35	0.001377	0.379300	0.009370	0.6814	0.3172	0.0078	0.9908
36	0.001604	0.340640	0.008730	0.7088	0.2896	0.0074	0.9910
37	0.001854	0.304750	0.009660	0.7353	0.2629	0.0083	0.9898
38	0.002089	0.274410	0.010370	0.7582	0.2397	0.0091	0.9889
39	0.002220	0.251820	0.010570	0.7756	0.2221	0.0093	0.9885
40	0.002331	0.237500	0.008190	0.7866	0.2111	0.0073	0.9904
41	0.002559	0.230650	0.008670	0.7919	0.2055	0.0077	0.9897
42	0.002869	0.229460	0.009600	0.7927	0.2044	0.0086	0.9886
43	0.003341	0.231850	0.010500	0.7905	0.2061	0.0093	0.9873
44	0.003679	0.236230	0.012440	0.7870	0.2094	0.0110	0.9853
45	0.003875	0.240450	0.011710	0.7834	0.2127	0.0104	0.9858
46	0.004171	0.242860	0.012200	0.7813	0.2145	0.0108	0.9851
47	0.004482	0.241200	0.013880	0.7825	0.2130	0.0123	0.9833
48	0.004900	0.234270	0.015370	0.7878	0.2073	0.0136	0.9815
49	0.005483	0.221660	0.017530	0.7976	0.1970	0.0156	0.9790
50	0.006406	0.204470	0.019970	0.8109	0.1827	0.0178	0.9758
51	0.007386	0.185040	0.019070	0.8260	0.1667	0.0172	0.9755
52	0.008007	0.165660	0.020620	0.8416	0.1504	0.0187	0.9733
53	0.008411	0.147960	0.023110	0.8564	0.1352	0.0211	0.9705
54	0.008972	0.132980	0.026080	0.8689	0.1221	0.0240	0.9671
55	0.009876	0.120890	0.028230	0.8787	0.1114	0.0260	0.9642
56	0.010748	0.111010	0.032840	0.8868	0.1025	0.0303	0.9590
57	0.011871	0.103250	0.036300	0.8928	0.0954	0.0335	0.9547
58	0.013511	0.097030	0.040470	0.8970	0.0896	0.0374	0.9492
59	0.015083	0.091630	0.046700	0.9006	0.0845	0.0430	0.9420
60	0.016280	0.086830	0.051160	0.9039	0.0800	0.0471	0.9367
61	0.017890	0.082600	0.052510	0.9062	0.0761	0.0483	0.9339
62	0.020671	0.078570	0.057910	0.9074	0.0721	0.0531	0.9264
63	0.023411	0.074720	0.066760	0.9086	0.0682	0.0610	0.9159
64	0.025597	0.070650	0.086570	0.9108	0.0639	0.0783	0.8964
65	0.028280	0.065750	0.109690	0.9133	0.0588	0.0982	0.8740
66	0.031012	0.063300	0.152780	0.9140	0.0555	0.1339	0.8356
67	0.033397	0.062050	0.592000	0.9217	0.0454	0.4333	0.5338
68	0.036293	0.053820	0.310740	0.9203	0.0440	0.2542	0.7101
69	0.039273	0.043810	0.387380	0.9267	0.0348	0.3076	0.6539
70	0.042486	0.044910	0.388590	0.9229	0.0355	0.3074	0.6510
71	0.046803	0.031620	0.293400	0.9282	0.0261	0.2417	0.7125
72	0.051322	0.039470	0.398030	0.9190	0.0309	0.3119	0.6381
73	0.055770	0.023390	0.356970	0.9271	0.0187	0.2851	0.6606
74	0.061222	0.030290	0.440420	0.9174	0.0232	0.3376	0.6030

Key: 1—inactive (outside labor force); 2—active (inside labor force).

Source: the rates are from Hoem and Fong (1976b, page 7).

The life history of the initial cohort is given in table 2. From a cohort of 100000 births, 97562 children survive to age 16. Up to this age they are all inactive. The transitions during single years of age ($x, x+1$) are computed by applying transition probabilities to the numbers of people at exact age x . For example, the number of people entering the labor force at age 20 is

$${}_1{}_{20}l_2(21) = p_{12}(20) \cdot {}_1l_1(20)$$

(60)

or

$$8355 = 0.3586 \times 23301 \text{ .}$$

(61)

The total number of people in the labor force at age 21 is

$${}_1l_2(21) = {}_1{}_{20}l_2(21) + {}_2{}_{20}l_2(21)$$

(62)

or

$$76780 = 8355 + 68424 \text{ .}$$

(63)

Table 3 gives the number of years spent in active and inactive life within the age interval x to $x+1$ per unit cohort of age 16. For example,

$${}_1{}_{16}L_2(20) = \frac{\frac{1}{2} [{}_1{}_{16}l_2(20) + {}_1{}_{16}l_2(21)]}{{}_1{}_{16}l_1(16)}$$

(64)

or

$$0.7720 = \frac{\frac{1}{2} (73853 + 76780)}{97562} \text{ .}$$

(65)

The fraction of a year spent in the labor force between ages 20 and 21 per unit cohort born is

$$0.7720 \frac{97562}{100000} = 0.7532 \text{ .}$$

(66)

The difference between 0.7720 and 0.7532 is due to mortality before the active ages.

An important statistic of the working-life table is the number of years in active life. Two approaches have been followed: the population-based measure of active life and the labor-force-based measure. The population-based measure is given in table 3. Since at age 16 everyone is inactive, the population-based measure is identical to the expression of the life expectancy by state at age 16. In other words, computing an increment-decrement life table by state at age 16 yields a population-based measure of the expected number of years in active and inactive life. The same measure is

Figure 1. Labor-force accession and separation schedules, Denmark, 1972-1974, males (adapted from Hoem and Fong, 1976a, page 24).

Table 2. Life history of the initial cohort.

Age	Deaths		Transitions				Number of people at exact age x		
	inactive	active	1 to 1	2 to 1	1 to 2	2 to 2	inactive	active	total
16	71	0	74410	0	23081	0	97562	0	97562
17	69	22	41926	2852	32414	20208	74410	23081	97491
18	54	63	24204	5153	20521	47405	44778	52622	97399
19	39	90	17122	6179	12196	61657	29357	67926	97282
20	28	90	14917	5338	8355	68424	23301	73853	97153
21	21	78	14091	5317	6144	71385	20255	76780	97035
22	18	73	13868	5147	5522	72308	19408	77528	96936
23	19	78	13612	4607	5384	73146	19015	77830	96845
24	18	80	12842	3731	5359	74718	18219	78529	96748
25	15	75	11208	2713	5349	77289	16573	80077	96650
26	13	74	9111	2169	4798	80395	13922	82639	96560
27	11	83	7171	1664	4098	83446	11280	85193	96473
28	9	90	5470	1365	3356	86089	8835	87544	96379
29	7	93	4146	1076	2681	88276	6835	89445	96279
30	6	97	3142	872	2075	89988	5222	90957	96179
31	5	110	2425	889	1584	91064	4014	92063	96077
32	4	122	2039	931	1271	91594	3315	92647	95962
33	4	119	1879	962	1087	91784	2970	92866	95835
34	4	117	1863	821	975	91934	2841	92871	95712
35	4	128	1829	728	851	92052	2684	92908	95592
36	4	149	1812	689	740	92065	2557	92904	95460
37	5	172	1839	773	658	91860	2502	92806	95307
38	5	193	1981	838	626	91487	2613	92518	95131
39	6	204	2187	859	626	91050	2819	92113	94932
40	7	213	2396	667	643	90796	3046	91676	94722
41	8	234	2426	706	629	90498	3063	91438	94501
42	9	261	2483	779	640	90087	3132	91128	94260
43	11	303	2579	847	672	89578	3262	90728	93990
44	13	331	2696	995	717	88924	3426	90250	93676
45	14	347	2891	929	785	88366	3691	89641	93332
46	16	371	2985	961	820	87819	3820	89151	92971
47	18	396	3087	1087	840	87156	3946	88639	92584
48	20	430	3288	1197	865	86369	4174	87996	92170
49	25	477	3577	1359	883	85399	4485	87235	91720
50	32	551	4003	1540	902	84191	4936	86282	91218
51	41	626	4578	1462	924	83005	5542	85093	90636
52	48	669	5083	1571	908	81689	6040	83929	89969
53	56	692	5699	1744	900	80161	6654	82597	89251
54	66	724	6468	1942	909	78395	7443	81061	88504
55	83	779	7389	2064	937	76461	8409	79304	87713
56	101	827	8383	2347	969	74224	9453	77398	86851
57	127	887	9580	2522	1024	71783	10730	75193	85923
58	162	977	10855	2721	1084	69108	12102	72807	84909
59	203	1051	12226	3022	1147	66120	13576	70193	83769
60	246	1086	13783	3169	1219	63012	15248	67267	82515
61	301	1139	15362	3105	1289	59987	16952	64231	81183
62	378	1254	16758	3256	1332	56766	18467	61276	79743
63	463	1344	18186	3542	1366	53211	20014	58097	78112
64	549	1379	19789	4274	1389	48923	21727	54577	76304
65	671	1403	21977	4939	1416	43970	24064	50312	74376
66	822	1386	24600	6077	1493	37923	26916	45386	72302
67	1008	1295	28277	17080	1393	21041	30678	39416	70094
68	1617	800	41743	5704	1997	15931	45357	22434	67791
69	1827	691	43969	5514	1650	11723	47447	17928	65375
70	2059	556	45667	4111	1758	8707	49483	13374	62857
71	2276	479	46204	2530	1297	7456	49777	10465	60242
72	2439	438	44788	2730	1507	5585	48734	8753	57487
73	2578	385	44052	2022	888	4686	47518	7093	54610
74	2737	331	42267	1881	1070	3361	46074	5573	51647

Key: 1—inactive; 2—active.

Table 3. Expected number of years lived within age interval x to $x + 1$ and beyond age x .

Age	Expected number of years lived by member of initial cohort					
	within age interval x to $x + 1$			beyond age x		
	inactive	active	total	inactive	active	total
16	0.8813	0.1189	0.9996	10.6390	42.0028	52.6414
17	0.6108	0.3880	0.9988	9.7648	41.9148	51.6796
18	0.3799	0.6178	0.9977	9.1621	41.5654	50.7274
19	0.2699	0.7266	0.9965	8.7921	40.9958	49.7879
20	0.2232	0.7720	0.9952	8.5327	40.3206	48.8533
21	0.2033	0.7908	0.9941	8.3187	39.5936	47.9124
22	0.1969	0.7962	0.9931	8.1226	38.8380	46.9606
23	0.1908	0.8013	0.9922	7.9320	38.0725	46.0044
24	0.1783	0.8128	0.9912	7.7474	37.3024	45.0498
25	0.1563	0.8339	0.9902	7.5753	36.5198	44.0950
26	0.1292	0.8601	0.9893	7.4244	35.7113	43.1357
27	0.1031	0.8853	0.9884	7.3005	34.8736	42.1741
28	0.0803	0.9071	0.9874	7.2033	34.0117	41.2150
29	0.0618	0.9246	0.9863	7.1294	33.1277	40.2571
30	0.0473	0.9380	0.9853	7.0741	32.2243	39.2984
31	0.0376	0.9466	0.9842	7.0336	31.3062	38.3398
32	0.0322	0.9507	0.9830	7.0038	30.3813	37.3851
33	0.0298	0.9519	0.9817	6.9803	29.4535	36.4338
34	0.0283	0.9521	0.9804	6.9589	28.5211	35.4800
35	0.0269	0.9523	0.9791	6.9388	27.5853	34.5241
36	0.0259	0.9518	0.9777	6.9209	26.6501	33.5710
37	0.0262	0.9498	0.9760	6.9055	25.7186	32.6240
38	0.0278	0.9462	0.9741	6.8914	24.7923	31.6837
39	0.0301	0.9419	0.9720	6.8772	23.8717	30.7489
40	0.0313	0.9385	0.9698	6.8615	22.9546	29.8161
41	0.0317	0.9356	0.9674	6.8452	22.0393	28.8845
42	0.0328	0.9320	0.9648	6.8299	21.1273	27.9572
43	0.0343	0.9275	0.9618	6.8155	20.2206	27.0361
44	0.0365	0.9219	0.9584	6.8026	19.3223	26.1250
45	0.0385	0.9163	0.9548	6.7896	18.4298	25.2194
46	0.0398	0.9112	0.9510	6.7755	17.5398	24.3154
47	0.0416	0.9052	0.9469	6.7619	16.6530	23.4149
48	0.0444	0.8980	0.9424	6.7482	15.7696	22.5178
49	0.0483	0.8893	0.9375	6.7342	14.8918	21.6260
50	0.0537	0.8783	0.9320	6.7196	14.0226	20.7421
51	0.0594	0.8662	0.9256	6.7049	13.1673	19.8722
52	0.0651	0.8534	0.9185	6.6903	12.3256	19.0158
53	0.0722	0.8387	0.9110	6.6730	11.4917	18.1647
54	0.0812	0.8219	0.9031	6.6497	10.6642	17.3139
55	0.0915	0.8031	0.8946	6.6192	9.8462	16.4654
56	0.1034	0.7820	0.8855	6.5821	9.0418	15.6239
57	0.1170	0.7585	0.8755	6.5358	8.2515	14.7873
58	0.1316	0.7329	0.8645	6.4794	7.4786	13.9580
59	0.1477	0.7045	0.8522	6.4142	6.7267	13.1410
60	0.1650	0.6739	0.8389	6.3371	5.9960	12.3331
61	0.1815	0.6432	0.8247	6.2428	5.2846	11.5273
62	0.1972	0.6118	0.8090	6.1334	4.5930	10.7264
63	0.2139	0.5775	0.7914	6.0152	3.9248	9.9400
64	0.2347	0.5376	0.7722	5.8841	3.2795	9.1636
65	0.2613	0.4904	0.7517	5.7289	2.6594	8.3882
66	0.2952	0.4346	0.7298	5.5406	2.0739	7.6145
67	0.3897	0.3170	0.7067	5.3044	1.5343	6.8386
68	0.4756	0.2069	0.6825	4.9237	1.1302	6.0540
69	0.4968	0.1604	0.6572	4.3959	0.8633	5.2592
70	0.5087	0.1222	0.6309	3.8010	0.6489	4.4499
71	0.5049	0.0985	0.6034	3.1421	0.4792	3.6213
72	0.4933	0.0812	0.5745	2.4359	0.3350	2.7709
73	0.4797	0.0649	0.5446	1.6830	0.2076	1.8905
74	0.4624	0.0513	0.5137	0.8735	0.0968	0.9703

Table 4. Number of years lived in each state between ages x and $x+1$.

Age	State at age x					
	inactive			active		
	inactive	active	total	inactive	active	total
17	0.7817	0.2178	0.9995	0.0618	0.9378	0.9995
18	0.7703	0.2291	0.9994	0.0490	0.9504	0.9994
19	0.7916	0.2077	0.9993	0.0455	0.9538	0.9993
20	0.8201	0.1793	0.9994	0.0361	0.9632	0.9994
21	0.8478	0.1516	0.9995	0.0346	0.9649	0.9995
22	0.8573	0.1423	0.9995	0.0332	0.9663	0.9995
23	0.8579	0.1416	0.9995	0.0296	0.9699	0.9995
24	0.8524	0.1471	0.9995	0.0238	0.9757	0.9995
25	0.8381	0.1614	0.9995	0.0169	0.9826	0.9995
26	0.8272	0.1723	0.9996	0.0131	0.9864	0.9996
27	0.8179	0.1816	0.9995	0.0098	0.9897	0.9995
28	0.8096	0.1899	0.9995	0.0078	0.9917	0.9995
29	0.8033	0.1961	0.9995	0.0060	0.9935	0.9995
30	0.8008	0.1986	0.9995	0.0048	0.9947	0.9995
31	0.8021	0.1973	0.9994	0.0048	0.9946	0.9994
32	0.8075	0.1918	0.9993	0.0050	0.9943	0.9993
33	0.8164	0.1830	0.9994	0.0052	0.9942	0.9994
34	0.8278	0.1715	0.9994	0.0044	0.9950	0.9994
35	0.8407	0.1586	0.9993	0.0039	0.9954	0.9993
36	0.8544	0.1448	0.9992	0.0037	0.9955	0.9992
37	0.8676	0.1314	0.9991	0.0042	0.9949	0.9991
38	0.8791	0.1199	0.9990	0.0045	0.9944	0.9990
39	0.8878	0.1111	0.9989	0.0047	0.9942	0.9989
40	0.8933	0.1055	0.9988	0.0036	0.9952	0.9988
41	0.8960	0.1028	0.9987	0.0039	0.9949	0.9987
42	0.8964	0.1022	0.9986	0.0043	0.9943	0.9986
43	0.8953	0.1031	0.9983	0.0047	0.9937	0.9983
44	0.8935	0.1047	0.9982	0.0055	0.9926	0.9982
45	0.8917	0.1064	0.9981	0.0052	0.9929	0.9981
46	0.8906	0.1073	0.9979	0.0054	0.9925	0.9979
47	0.8912	0.1065	0.9978	0.0061	0.9916	0.9978
48	0.8939	0.1036	0.9976	0.0068	0.9908	0.9976
49	0.8988	0.0985	0.9973	0.0078	0.9895	0.9973
50	0.9054	0.0914	0.9968	0.0089	0.9879	0.9968
51	0.9130	0.0834	0.9963	0.0086	0.9877	0.9963
52	0.9208	0.0752	0.9960	0.0094	0.9866	0.9960
53	0.9282	0.0676	0.9958	0.0106	0.9852	0.9958
54	0.9345	0.0611	0.9955	0.0120	0.9836	0.9955
55	0.9394	0.0557	0.9951	0.0130	0.9821	0.9951
56	0.9434	0.0512	0.9946	0.0152	0.9795	0.9946
57	0.9464	0.0477	0.9941	0.0168	0.9773	0.9941
58	0.9485	0.0448	0.9933	0.0187	0.9746	0.9933
59	0.9503	0.0422	0.9925	0.0215	0.9710	0.9925
60	0.9519	0.0400	0.9919	0.0236	0.9684	0.9919
61	0.9531	0.0380	0.9911	0.0242	0.9670	0.9911
62	0.9537	0.0360	0.9898	0.0266	0.9632	0.9898
63	0.9543	0.0341	0.9884	0.0305	0.9580	0.9884
64	0.9554	0.0320	0.9874	0.0392	0.9482	0.9874
65	0.9566	0.0294	0.9860	0.0491	0.9370	0.9860
66	0.9570	0.0277	0.9847	0.0670	0.9178	0.9847
67	0.9609	0.0227	0.9836	0.2167	0.7669	0.9836
68	0.9602	0.0220	0.9822	0.1271	0.8550	0.9822
69	0.9634	0.0174	0.9807	0.1538	0.8270	0.9807
70	0.9614	0.0178	0.9792	0.1537	0.8255	0.9792
71	0.9641	0.0130	0.9771	0.1209	0.8563	0.9771
72	0.9595	0.0155	0.9750	0.1559	0.8190	0.9750
73	0.9635	0.0093	0.9729	0.1426	0.8303	0.9729
74	0.9587	0.0116	0.9703	0.1688	0.8015	0.9703

Table 5. Expectations of inactive and active life by state at age *x*.

Age	State at age <i>x</i>					
	inactive			active		
	inactive	active	total	inactive	active	total
17	10·092	41·588	51·680	8·711	42·969	51·680
18	9·974	40·753	50·728	8·471	42·256	50·728
19	10·025	39·763	49·788	8·259	41·529	49·788
20	10·108	38·745	48·853	8·036	40·818	48·853
21	10·113	37·799	47·912	7·845	40·067	47·912
22	9·980	36·980	46·961	7·658	39·303	46·960
23	9·790	36·214	46·004	7·478	38·526	46·004
24	9·582	35·468	45·050	7·322	37·726	45·050
25	9·379	34·716	44·095	7·202	36·593	44·095
26	9·231	33·905	43·136	7·120	36·016	43·136
27	9·123	33·051	42·174	7·059	35·115	42·174
28	9·054	32·161	41·215	7·016	34·198	41·215
29	9·032	31·225	40·257	6·984	33·273	40·257
30	9·063	30·235	39·298	6·960	32·338	39·298
31	9·150	29·190	38·340	6·941	31·398	38·340
32	9·295	28·090	37·385	6·922	30·463	37·385
33	9·495	26·938	36·434	6·900	29·534	36·434
34	9·741	25·739	35·480	6·874	28·606	35·480
35	10·009	24·515	34·524	6·850	27·674	34·524
36	10·276	23·294	33·571	6·828	26·742	33·571
37	10·511	22·113	32·624	6·808	25·816	32·624
38	10·690	20·993	31·684	6·784	24·900	31·684
39	10·804	19·945	30·749	6·757	23·992	30·749
40	10·857	18·959	29·816	6·729	23·387	29·816
41	10·866	18·018	28·884	6·710	22·174	28·884
42	10·853	17·104	27·957	6·692	21·266	27·957
43	10·840	16·196	27·036	6·671	20·365	27·036
44	10·846	15·278	26·125	6·649	19·476	26·125
45	10·886	14·333	25·219	6·621	18·548	25·219
46	10·967	13·349	24·315	6·596	17·719	24·315
47	11·093	12·322	23·415	6·569	16·846	23·415
48	11·258	11·260	22·518	6·534	15·984	22·518
49	11·447	10·178	21·626	6·492	15·134	21·626
50	11·636	9·106	20·742	6·438	14·304	20·742
51	11·797	8·075	19·872	6·373	13·499	19·872
52	11·903	7·113	19·016	6·315	12·702	19·016
53	11·932	6·232	18·165	6·249	11·915	18·165
54	11·875	5·439	17·314	6·170	11·144	17·314
55	11·736	4·729	16·465	6·077	10·389	16·465
56	11·529	4·095	15·624	5·978	9·646	15·624
57	11·259	3·528	14·787	5·862	8·926	14·787
58	10·938	3·020	13·958	5·738	8·220	13·958
59	10·576	2·564	13·141	5·609	7·532	13·141
60	10·176	2·157	12·333	5·467	6·866	12·333
61	9·734	1·793	11·527	5·321	6·206	11·527
62	9·254	1·472	10·726	5·193	5·533	10·726
63	8·744	1·196	9·940	5·075	4·865	9·940
64	8·200	0·963	9·164	4·962	4·202	9·164
65	7·616	0·772	8·388	4·826	3·562	8·388
66	6·990	0·625	7·614	4·681	2·933	7·614
67	6·327	0·512	6·839	4·508	2·331	6·839
68	5·652	0·402	6·054	3·452	2·602	6·054
69	4·959	0·300	5·259	2·906	2·352	5·259
70	4·228	0·222	4·450	2·221	2·229	4·450
71	3·481	0·141	3·621	1·532	2·089	3·621
72	2·681	0·090	2·771	1·270	1·700	2·771
73	1·855	0·035	1·890	0·527	1·363	1·890
74	0·959	0·012	0·970	0·169	0·801	0·970

obtained by computing the life expectancy at birth. A 16 year old is expected to spend about 42 years in active life and about 11 years in inactive life. At birth a baby is expected to spend $42 \times 0.97562 = 40.98$ years in the labor force. As before, the difference is due to mortality at ages below 16 years. The population-based measure of the expected remaining numbers of active years of a 20 year old is

$${}_2e_2(20) = \sum_{y=20}^{74} \frac{{}_2L_2(y)}{{}_2\dot{L}_2(20)}, \quad (67)$$

where ${}_2\dot{L}_2(20) = {}_2\dot{L}_2(20)/{}_2\dot{L}_2(16) = \frac{97153}{97562} = 0.99581$. Hence ${}_2e_2(20)$ is

$$40.321 = \frac{40.1518}{0.99581}. \quad (68)$$

To obtain numerical values for the labor-force-based measure of active life, the 'multiregional' life table 'by place of residence' or current state was calculated. The life history of the population by current state is identical to table 2. This is as expected since table 2 also does not distinguish between different cohorts (taking the cohort of the inactive population of age 16 implies the total population, because at age 16 everyone is inactive). The fraction of a year spent inside and outside the labor force between ages x and $x+1$ by a person who is in (or outside) the labor force *at exact age* x is given in table 4. For example, a male inactive person aged 20 is expected to spend ${}_1{}_{20}L_2(20) = 0.1793$ years in the labor force and to spend an average of ${}_1{}_{20}L_1(20) = 0.8201$ years in inactive life before reaching the age of 21. On the other hand, a person already in the labor force at age 20 spends an average of ${}_2{}_{20}L_2(20) = 0.9632$ years in the labor force and ${}_2{}_{20}L_1(20) = 0.0361$ years in inactive life before reaching 21 years. Table 4 gives the average time spent in each state during the following year per active and inactive person of exact age x .

Instead of considering the average time spent in each state during one-year intervals, one may calculate the time spent in each state beyond age x . The result is represented in table 5. It gives the remaining lifetime inside and outside the labor force per active and inactive person of exact age x . The right-hand side of the table shows the labor-force-based measures of active and inactive life. For example, an active person aged 20 years may expect to spend 40.818 years in the labor force and 8.036 years in inactive life. The left-hand side of the table contains the inactive-life-based measures of active and inactive life. A 20-year-old person who is inactive is expected to spend 38.745 and 10.108 years in active and inactive life respectively. Table 5 begins with age 17 because no person of age 16 is in the labor force and therefore a labor-force-based measure of active life below this age is not meaningful.

5 Application: labor-force projections

Traditionally, labor-force projections are made by applying trends of LFP rates to population projections by sex and age. The conventional procedure focuses on the *stock* of the labor force. In order to include people who enter and leave the labor force at each age, Cohn et al (1974) developed a model denoted here as the CNN model. However, new entrants into the labor force are not generated by the model but must be projected separately. A model that projects the active and inactive population simultaneously and that accounts for the differences and the interactions between these two states is proposed in subsection 5.3. It is similar to the multi-regional demographic growth model (Rogers, 1975, chapter 5).

5.1 Traditional techniques

The general approach to labor-force projections is to project the population by age and sex and to apply trend-adjusted LFP rates to the projected population. Let $K(t)$

denote the age composition of the population (by sex), $G(t)$ the growth matrix, and $W(t)$ a diagonal matrix of age-specific LFP rates. The labor force by age group at time $t+1$ then becomes

$$E(t+1) = W(t+1)K(t+1) = W(t+1)G(t)K(t) . \quad (69)$$

If $W(t)$ and $G(t)$ remain constant, we have

$$E(t+1) = WGK(t) = HK(t) . \quad (70)$$

The matrix H is of the same form as G .

If the rate of change of labor-force participation rates is fixed, then $W(t) = D^t W(0)$, where D is a diagonal matrix whose nonzero elements are equal to one plus the rate of change of the LFP rate, and $W(0)$ is the matrix of LFP rates in the base year (Fullerton and Prescott, 1975, page 69). A logistic curve has been used by Im and Ramachandran (1961).

This approach is also being used by the US Bureau of Labor Statistics (see, for example, Johnston, 1973a, page 11; 1973b, page 15) and by the Organisation for European Economic Cooperation (OEEC, 1961; 1966). Current labor-force projections reflect changes in the age composition of the population and changes in the age-specific LFP rates. Johnston clearly states which of the two components is more important:

“The predominant factor in these projections is the anticipated changes in the size and age-sex distribution of the population; projected changes in participation rates play a relatively minor role” (Johnston, 1973b, page 17).

As far as the LFP rates are concerned, the most frequent assumptions made are (Johnston, 1973a, page 11; Rosenblum, 1972):

- (a) full employment (a generally favorable demand situation, or the demand for labor is completely elastic);
- (b) no significant change in the size of the armed forces;
- (c) social and political stability;
- (d) continuation of education trends; and
- (e) no change in the definition of ‘labor force’, ‘employment’, or ‘unemployment’.

5.2 The CNN model

The method of applying age-specific labor-force participation rates to the projected population gives an estimate of the future population in active life. It does not distinguish between new entrants and people leaving the labor force. A projection procedure which explicitly considers new entrants and leavers was proposed by Cohn et al (1974) for the forecasting of the aggregate supply of coalminers. The model considers five-year age groups and projection intervals of five years. The procedure consists of two parts. The first part is the estimation of the expected labor force for each of the projection years, disregarding new entrants, and the second part is the estimation of the expected number of entrants into the labor force by age.

To estimate the expected labor force, disregarding new entrants, they use the concept of a labor-force retention rate, which is one minus the total separation rate. The expected number of people active at time t and aged x to $x+4$ who will still be active five years later is

$$E_s^{(t+1)}(x+5) = [1 - M^w(x)]E^{(t)}(x) , \quad (71)$$

where $E_s^{(t)}(x)$ is the surviving active population in age group x to $x+4$ at time t , $E^{(t)}(x)$ is the total labor force aged x to $x+4$ at time t , and $[1 - M^w(x)]$ is the labor-force retention rate. Writing equation (70) for several age groups, we have the matrix

expression

$$\begin{bmatrix} E^{(t+1)}(\bar{\alpha}) \\ E^{(t+1)}(\bar{\alpha}+5) \\ \vdots \\ E^{(t+1)}(x-5) \\ E^{(t+1)}(x) \\ \vdots \\ E^{(t+1)}(\bar{\beta}-5) \\ E^{(t+1)}(\bar{\beta}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ R^w(\bar{\alpha}) & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & R^w(x-5) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & R^w(\bar{\beta}-5) & 0 \end{bmatrix} \begin{bmatrix} E^{(t)}(\bar{\alpha}) \\ E^{(t)}(\bar{\alpha}+5) \\ \vdots \\ E^{(t)}(x-5) \\ E^{(t)}(x) \\ \vdots \\ E^{(t)}(\bar{\beta}-5) \\ E^{(t)}(\bar{\beta}) \end{bmatrix}, \quad (72)$$

where $R^w(x) = [1 - M^w(x)]$. Equation (72) may be written more concisely as

$$E_s^{(t+1)} = \mathbf{R}E^{(t)}. \quad (73)$$

The proposed method for estimating the expected number of entrants into the labor force by age is based on two premises. First, the number of new entries in the five-year period around the base year is assumed to be equal to the difference between the actual labor force at the end of the interval and the expected labor force at the end of the interval (calculated by the retention-rate method). Second, it is assumed that these entry levels by age will remain constant: that is, at each projection interval the same number of people enter the labor force and their age composition is fixed. The number of new entrants by age are computed as follows:

$$E_n^{(t+1)}(x) = E^{(t+1)}(x), \quad \text{for } x = \bar{\alpha}, \quad (74)$$

$$E_n^{(t+1)}(x) = E^{(t+1)}(x) - [1 - M^w(x-5)]E^{(t)}(x-5), \quad \text{for } x > \bar{\alpha}. \quad (75)$$

In other words, the labor force in age group $\bar{\alpha}$ at time $t+1$ is completely made up of new entrants (between t and $t+1$). The labor force in age group $(x, x+4)$ for $x > \bar{\alpha}$ at time t is composed of people who were in the labor force at t and survived, plus new entrants between t and $t+1$:

$$E^{(t+1)}(x) = E_s^{(t+1)}(x) + E_n^{(t+1)}(x) = [1 - M^w(x-5)]E^{(t)}(x-5) + E_n^{(t+1)}(x), \quad \text{for } x > \bar{\alpha}. \quad (76)$$

Note that, if $E^{(t+1)}(x)$ is less than $E^{(t)}(x)$, then $E^{(t+1)}(x) = 0$. The total labor force at time $t+1$ therefore is

$$E^{(t+1)} = \mathbf{R}E^{(t)} + E_n^{(t+1)}, \quad (77)$$

where $E_n^{(t+1)}$ is the vector of new entrants during the time interval t to $t+1$.

5.3 The two-state projection model

The labor-force projection model proposed in this section proceeds from the analysis of increment-decrement tables of working life and uses findings of multiregional demography (Rogers, 1975). The active and inactive populations by age (and sex) are projected simultaneously. This method enables one to consider mortality and fertility differences between the active and inactive population and to examine the interactions between these subsets of the population.

The necessary data for labor-force projection by this method consist of:

- the active and inactive population by age in the base year (all ages);
- the age-specific rates of mortality both of the active and of the inactive populations, and rates of accession and separation;
- the age-specific birth rates for the active and inactive populations.

The model is presented for the case of a single-sex population: the extension to a two-sex population is straightforward. The active and inactive population by age at time t is denoted by the vector $K(t)$.

$$K(t) = \begin{bmatrix} K^{(t)}(0) \\ K^{(t)}(1) \\ \vdots \\ K^{(t)}(x) \\ \vdots \\ K^{(t)}(z) \end{bmatrix}, \quad \text{where } K^{(t)}(x) = \begin{bmatrix} K_1^{(t)}(x) \\ K_2^{(t)}(x) \end{bmatrix}. \quad (78)$$

An element $K_i^{(t)}(x)$ denotes the number of people aged x to $x+1$ in state i at time t ($i = 1$ for the inactive population, $i = 2$ for the active population). At ages below $\bar{\alpha}$ and above $\bar{\beta}$, $K_2^{(t)}(x)$ will be zero. Of the people aged x to $x+1$ in the labor force at time t , some may remain in the labor force during the time period $(t, t+1)$, some may become inactive, and some may die. On the other hand, the people aged $x+1$ to $x+2$ in the labor force at time $t+1$ are composed of active-population survivors at time t and of new entrants into the labor force:

$$K_2^{(t+1)}(x+1) = s_{22}(x)K_2^{(t)}(x) + s_{12}(x)K_1^{(t)}(x), \quad (79)$$

where $s_{22}(x)$ and $s_{12}(x)$ are survivorship proportions, which may be derived as part of the working-life table or computed directly from age-specific rates of mortality, accession, and separation:

$$S(x) = [I + \frac{1}{2}M(x+1)]^{-1}[I - \frac{1}{2}M(x)]. \quad (80)$$

The inactive population aged $x+1$ to $x+2$ at time $t+1$ is

$$K_1^{(t+1)}(x+1) = s_{11}(x)K_1^{(t)}(x) + s_{21}(x)K_2^{(t)}(x). \quad (81)$$

Equations (79) and (81) may be combined into the following matrix expression

$$K^{(t+1)}(x+1) = S(x)K^{(t)}(x). \quad (82)$$

The number of children in the first age group (from 0 to 1) at time $t+1$ is given by the surviving births during the interval $(t, t+1)$. Let $F_1(x)$ and $F_2(x)$ be the age-specific annual birth rates of a member of the labor force and of the inactive population respectively. It is realistic to allow for different fertility schedules for the active and the inactive populations. Assuming a uniform distribution of births over the time interval, the number of births between t and $t+1$ to mothers aged x to $x+1$ at time t is equal to

$$\sum_{i=1}^2 \frac{1}{2} [F_i(x)K_i^{(t)}(x) + F_i(x+1)K_i^{(t+1)}(x+1)]. \quad (83)$$

Not all of these babies will survive to become members of the first age group at time $t+1$. (However, all the survivors will be in state 1, that is, inactive.) Assuming that infant mortality is independent of the activity status of the parent, the number of children aged 0 to 1 years at time $t+1$ is given by

$$K_1^{(t+1)}(0) = \sum_x \frac{1}{2} \hat{p} \sum_{i=1}^2 [F_i(x)K_i^{(t)}(x) + F_i(x+1)K_i^{(t+1)}(x+1)] \quad (84)$$

and

$$K_2^{(t+1)}(0) = 0, \quad (85)$$

where $\hat{p}$ denotes the proportion of babies born during the interval $(t, t+1)$ that will survive to time $t+1$. In matrix notation we may write

$$K^{(t+1)}(0) = \sum_x \frac{1}{2} \hat{\mathbf{P}} [\mathbf{F}(x) K^{(t)}(x) + \mathbf{F}(x+1) K^{(t+1)}(x+1)] , \quad (86)$$

where

$$\mathbf{F}(x) = \begin{bmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{P}} = \begin{bmatrix} \hat{p} & \hat{p} \\ 0 & 0 \end{bmatrix} . \quad (87)$$

Equation (86) may be reformulated as follows:

$$K^{(t+1)}(0) = \sum_x \frac{1}{2} \hat{\mathbf{P}} [\mathbf{F}(x) + \mathbf{F}(x+1) \mathbf{S}(x)] K^{(t)}(x) = \sum_x \mathbf{B}(x) K^{(t)}(x) . \quad (88)$$

Equations (82) and (88) combined constitute the complete growth model of the active and inactive populations:

$$\begin{bmatrix} K^{(t+1)}(0) \\ K^{(t+1)}(1) \\ K^{(t+1)}(2) \\ \vdots \\ K^{(t+1)}(z) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & \mathbf{B}(\hat{\alpha}) & \dots & \mathbf{B}(\hat{\beta}) & 0 & \dots & 0 & 0 \\ \mathbf{S}(0) & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbf{S}(1) & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \mathbf{S}(z-1) & 0 \end{bmatrix} \begin{bmatrix} K^{(t)}(0) \\ K^{(t)}(1) \\ K^{(t)}(2) \\ \vdots \\ K^{(t)}(z) \end{bmatrix} \quad (89)$$

or

$$K(t+1) = \mathbf{G}K(t) . \quad (90)$$

The ages $\hat{\alpha}$ and $\hat{\beta}$ are respectively the lowest and highest ages of the reproductive period.

6 Conclusion

The purpose of this paper was to illustrate how the mathematical theory of multi-regional or multistate demography may be applied to calculate tables of working life *which do not rely on restrictive assumptions* such as the unimodality of the LFP curve, entrance into the labor force only at ages before the peak and retirement only at ages after the peak, and independence of mortality from the labor-force status. These assumptions are implicit in working-life tables that have been published to date in a large number of countries.

The fundamental difference between the new technique and the conventional method is its focus on *flows* instead of *stocks*. In the conventional method, only changes in the stock (net flows) of the active population by age are considered. The new technique focuses on gross flows in and out of the labor force at each age.

A second objective of the paper was to contribute to a theoretical analysis of multistate life-table construction without going into the details of movement versus transition approaches. All life-table functions, including the life-table rates, were derived directly from the Kolmogorov equation, and an attempt was made to state all the necessary assumptions explicitly.

A drawback of the new method for the calculation of working-life tables is the large data requirement. Today most countries do not publish the data required to build increment-decrement tables of working life. Therefore the design of better methods for labor-force analysis should be accompanied by greater attention to techniques for estimating missing data. Here, too, multiregional demography and migration research may contribute. Procedures developed to estimate age-specific migration flows, by origin and destination, from incomplete data may be applied to infer age-specific gross

flows in and out of the labor force, that is, between the inactive and active states (Rogers et al, 1977; Rees, 1977; Willekens et al, 1979).

It has been an important recent observation that the applicability of multiregional demography is not limited to the study of a system of regions but extends to the analysis of any system of states or categories of age-specific populations (marital status, health status, etc) for which increment-decrement life tables may be developed. And, in addition to life-table construction, multiregional demography also may be used for the development of better models of labor-force projection, such as the two-state projection model proposed in section 5.3 of this paper.

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References

- Bowen W, Finegan A, 1969 *The Economics of Labor Force Participation* (Princeton University Press, Princeton, NJ)
- Brauer F, Nohel J, Schneider H, 1970 *Linear Mathematics. An Introduction to Linear Algebra and Linear Differential Equations* (Benjamin, New York)
- Cohn E, Nelson J, Neumann G, 1974 "Forecasting aggregate supply of coal miners" *Socio-Economic Planning Sciences* 8 293-299
- Durand J D, 1948 *The Labor Force in the United States, 1890-1960* (Social Science Research Council, New York)
- Durand J D, 1975 *The Labor Force in Economic Development. A Comparison of International Census Data, 1946-1966* (Princeton University Press, Princeton, NJ)
- Durand J D, Miller A, 1968 *Methods of Analyzing Census Data on Economic Activities of the Population ST/SOA/Series A/43* (United Nations, New York)
- Farooq G, 1975 *Dimensions and Structure of Labour Force in Relation to Economic Development. A Comparative Study of Pakistan and Bangladesh* (Pakistan Institute of Development Economics, Islamabad, Pakistan)
- Fullerton H H, 1971 "A table of expected working life for men, 1968" *Monthly Labor Review* 94 (June) 49-55
- Fullerton H H, 1972 "A new type of working life table for men" *Monthly Labor Review* 95 (July) 20-27
- Fullerton H H, Byrne J J, 1976 "Length of working life for men and women, 1970" *Monthly Labor Review* 99 (February) 31-35
- Fullerton H H, Prescott J R, 1975 *An Economic Simulation Model for Regional Development Planning* (Ann Arbor Science Publishers, Ann Arbor, Mich.)
- Gantmacher F R, 1959 *The Theory of Matrices* 2 volumes (Chelsea, New York)
- Garfinkle S, 1967 "The lengthening of working life and its implications" in *World Population Conference, 1965, Volume IV* (United Nations, New York) pp 277-282
- Hoem J, Fong M, 1976a "A Markov chain model of working life tables: a new method for the construction of tables of working life" WP-2, Laboratory of Actuarial Mathematics, University of Copenhagen, Copenhagen
- Hoem J, Fong M, 1976b "A Markov chain model of working life tables: illustrative tables based on Danish labor force surveys, 1972-1974" Supplement to WP-2, Laboratory of Actuarial Mathematics, University of Copenhagen, Copenhagen
- Im T B, Ramachandran K V, 1961 *Labor Force Projections for the Republic of Korea, 1960-1975* (United Nations Demographic Training and Research Center, Bombay, India)
- Johnston D, 1973a "The U.S. labor force: projections to 1990" *Monthly Labor Review* 96 (July) 3-13
- Johnston D, 1973b "The United States economy in 1985: population and labor force projections" *Monthly Labor Review* 96 (December) 8-17
- Kpedekpo G, 1969 "Working life tables for males in Ghana, 1960" *Journal of the American Statistical Association* 64 102-110
- Kpedekpo G, 1974 "Methods of manpower calculations" in *Population in African Development. Volume 2* Ed. P Cantrelle (Éditions Ordina, Dalhain, Belgium)
- Krishnamoorthy S, 1979 "Classical approach to increment-decrement life tables: an application to the study of marital status of United States females, 1970" *Mathematical Biosciences* 44 139-154

- Ledent J, 1978 "Some methodological and empirical consideration in the construction of increment-decrement life tables" RM-78-25, International Institute for Applied Systems Analysis, Laxenburg, Austria
- Ledent J, 1980 "Multistate life tables: movement versus transition perspectives" *Environment and Planning A* 12 533-562 (in this issue)
- OEEC, 1961 *Demographic Trends in Western Europe and in the United States, 1956-1976* (Organisation for European Economic Cooperation, Paris)
- OEEC, 1966 *Demographic Trends in Western Europe and North America, 1965-1980* (Organisation for European Economic Cooperation, Paris)
- Rees P H, 1977 "The measurement of migration, from census data and other sources" *Environment and Planning A* 9 247-272
- Rogers A, 1973 "The multiregional life table" *Journal of Mathematical Sociology* 3 127-137
- Rogers A, 1975 *Introduction to Multiregional Mathematical Demography* (John Wiley, New York)
- Rogers A, Ledent J, 1976 "Increment-decrement life tables: a comment" *Demography* 13 287-290
- Rogers A, Raquillet R, Castro L J, 1977 "Model migration schedules and their applications" *Environment and Planning A* 10 475-502; also available as RM-77-57, International Institute for Applied Systems Analysis, Laxenburg, Austria
- Rosenblum M, 1972 "On the accuracy of labor force projections" *Monthly Labor Review* 95 22-29
- Schoen R, 1975 "Constructing increment-decrement life tables" *Demography* 12 313-324
- Smith S J, 1977 "Liability cases and the use of working life tables in court" paper presented at the 1977 Annual Meeting of the Population Association of America, St Louis, Mo, 21-23 April
- United Nations, 1973 *The Determinants and Consequences of Population Trends: New Summary of Findings on Interaction Between Demographic, Economic and Social Factors, Volume 1* (United Nations, New York)
- Willekens F J, 1977 "Sensitivity analysis in multiregional demographic growth models" *Environment and Planning A* 9 653-674
- Willekens F J, Rogers A, 1978 "Spatial population analysis: methods and computer programs" RR-78-18, International Institute for Applied Systems Analysis, Laxenburg, Austria
- Willekens F J, Pör A, Raquillet R, 1979 "Entropy, multiproportional, and quadratic techniques for inferring detailed migration patterns from aggregate data. Mathematical theories, algorithms, applications and computer programs" WP-79-88, International Institute for Applied Systems Analysis, Laxenburg, Austria
- Wolfbein S, 1949 "The length of working life" *Population Studies* 3 286-294
- Wolovich W, 1974 *Linear Multivariable Systems* (Springer, New York)

APPENDIX

This appendix gives the solution to equations (15) and (16), derived by Hoem and Fong (1976a, pages 68-70), assuming no mortality or equal mortality in both states.

No mortality

Without mortality [$\mu_{is}(x) = 0$, $i = 1, 2$], all people in state i at age y will be either in state i or in state j at age x ($x = y + n$); $i, j = 1, 2$). Therefore ${}_{iy}\bar{l}_i(x) = 1 - {}_{iy}\bar{l}_j(x)$, $i \neq j$, for a unit cohort, where the bar denotes the absence of mortality. Introducing this in equation (15) gives

$$\frac{d}{dx} {}_{iy}\bar{l}_2(x) = [1 - {}_{iy}\bar{l}_2(x)]\mu_{12}(x) - {}_{iy}\bar{l}_2(x)\mu_{21}(x), \quad (A1)$$

or equivalently

$$\frac{d}{dx} {}_{iy}\bar{l}_2(x) = \mu_{12}(x) - \gamma(x){}_{iy}\bar{l}_2(x), \quad (A2)$$

where $\gamma(x) = \mu_{21}(x) + \mu_{12}(x)$. The solution to equation (A2) is

$${}_{iy}\bar{l}_2(x) = \int_0^n \mu_{21}(y+t) \exp\left[-\int_t^n \gamma(y+u) du\right] dt. \quad (A3)$$

This formula is numerically evaluated by replacing the continuous functions $\mu_{21}(y+t)$ and $\gamma(y+t)$ by step functions. If $\mu_{21}(y+t) = \bar{\mu}_{21}(y)$ and $\gamma(y+t) = \bar{\gamma}(y)$ in the interval from y to $x = y+n$, then, since

$$\begin{aligned} \int_0^n \exp\left[-a \int_s^n du\right] ds &= \int_0^n \exp[-a(n-s)] ds = \int_0^n \exp(-an) \exp(as) ds \\ &= \exp(-an) \int_0^n \exp(as) ds = \exp(-an) \left[\frac{1}{a} \exp(an) - \frac{1}{a} \exp(0) \right] \\ &= \frac{1}{a} [1 - \exp(-an)] , \end{aligned} \tag{A4}$$

one gets

$${}_y\bar{l}_2(x) = \frac{\bar{\mu}_{21}(y)}{\bar{\gamma}(y)} \left\{ 1 - \exp[-n\bar{\gamma}(y)] \right\} . \tag{A5}$$

Equal mortality

In the case of equal mortality, the differential equation describing the dynamics of mortality is, for example,

$$\frac{d}{dx} {}_y l_\delta(x) = [1 - {}_y l_\delta(x)] \mu_\delta(x) . \tag{A6}$$

Note that for differential mortality in both states, equation (A6) does not hold; if mortality depends on the state (active/inactive), one must consider the state of the person at exact age x . The variable ${}_y l_\delta(x)$ measures the probability that a person who is inactive at age y will die before reaching the age x . Since mortality is equal in both states, it is not important where the person will be at the time of death.

The solution to equation (A6) is

$${}_y l_\delta(x) = 1 - \exp\left[-\int_0^n \mu_\delta(y+t) dt\right] = {}_y q(x) . \tag{A7}$$

Note that the solution is independent of the state at age y . The quantity ${}_y q(x)$ measures the probability of dying between ages y and $x = y+n$.

To derive an expression for ${}_y l_2(x)$ in the case of equal mortality, the equation

$${}_y l_1(x) = 1 - {}_y q(x) - {}_y l_2(x) \tag{A8}$$

is introduced into equation (15), which is then solved for ${}_y l_2(x)$:

$${}_y l_2(x) = {}_y \bar{l}_2(x) [1 - {}_y q(x)] . \tag{A9}$$

Hence the probability of changing states in the case of equal mortality is the product of the probability in the case of no mortality and the survivorship probability, $1 - {}_y q(x)$. In general we may write

$${}_y l_j(x) = {}_y \bar{l}_j(x) [1 - {}_y q(x)] . \tag{A10}$$

This formula separates the mortality and the mobility effects.

Note that this approach to solving equations (15) and (16) differs from that in the main body of the paper, when constant instantaneous rates in the interval $(y, y+n)$ were not assumed. These differences partly explain the deviations between Hoem and Fong's (1976a) results and the ones listed in this paper. The deviations are not, however, substantial.