A dynamic linear-programming approach to the planning of national settlement systems

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Abstract. In this paper the problem of planning human-settlement systems (HSS) is formulated in a dynamic linear-programming (DLP) framework. Such large time-dependent problems are treated both by optimal-control and linear-programming techniques. A multiregional population-growth model forms the state equation of the DLP problem. Budget, resource, and quality-of-life constraints are considered. The paper describes the formalization of the HSS planning problem and suggests methods for its solution.

1 Introduction

The development of human-settlement systems is becoming a public concern in most countries. Nations all over the world are adopting policies to guide the growth and the distribution of their populations (for some details, see Willekens, 1976a). This trend toward explicit national settlement policies is enhanced by the realization that land and environment are not free goods; they are scarce resources to be conserved. Settlement planning must elaborate control policies of population distribution over space and/or time in order to achieve desirable socioeconomic goals (conservation of the environment, economic efficiency, etc), and must take into account a large number of factors and constraints such as total population; age and sex structure; birth, death, and migration rates; scarceness of resources; and educational constraints. An effective way of developing optimal decisions, when a very large number of variables and constraints are involved, is by applying mathematical programming. Of the available optimization methods, linear programming is most successful in dealing with large static problems. Dynamic decision problems, on the other hand, have been treated by using optimal-control theory. National settlement systems are large in scale and dynamic in nature, and can be studied as dynamic optimization problems, more particularly as dynamic linear-programming (DLP) problems, which comprise both linear-programming (LP) and control-theory methods (Propoi, 1976).

The purpose of this paper is to discuss the applicability of the DLP approach to national settlement-system planning. It is the third IIASA paper on the application of specific mathematical-programming methods to settlement-system planning (Mehra, 1975; Evtushenko and MacKinnon, 1976), and consists of two parts—the first part describes DLP models of national settlement-system planning, and the second part is devoted to the application of DLP theory and methods in the solution of these models.

2 The planning problem

The purpose of this section is to describe in some detail the problem of national settlement-system planning. The models envisaged are in the format of a DLP problem. A DLP problem consists of three components: the state equations, the constraints imposed on the system variables, and the performance index (objective function). In DLP, all these equations and inequalities are linear in the state and the control variables.

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2.1 The state equations

The state equations describe the development of the multiregional population system over time. This development is determined by internal systems dynamics and by policy interventions. The internal dynamics are represented by the system's 'laws of motion'. External intervention will change the motion of the system, but the degree and the direction of the disturbance depend on the dynamic characteristics of the system.

To avoid counteractive and undesired effects of a settlement policy, we need to understand the internal dynamics governing a multiregional population system, that is, the behavior of the system over time before applying control to it. The mechanisms of demographic growth and intervention have been considered by Rogers (1968; 1971; 1975). They are explored further in this paper.

To transform the growth model into a policy model we add a sequence of vectors describing control actions distributed over time and space. A control vector defines the instruments of population-distribution policy. Population distribution does not occur in a vacuum; it is subordinate to social and economic policies. Frequently the goals of population redistribution are environmental and economic in nature. To achieve these nondemographic goals, use is made of nondemographic, economic, and legal instruments. Although the focus is on population and its distribution, policy implementation requires the consideration of socioeconomic factors. Demometrics is the study of this interdependence between spatial population growth and the socioeconomic system. This paper shows how demometrics might contribute to the formulation of models of national settlement-system planning by describing not only the internal system dynamics but also the influence of external intervention on the system.

Formally, the state equations appear as linear heterogenous equations. The homogenous part of the equation system describes the behavior of the system undisturbed by outside influences. (This behavior is defined by a multiregional demographic-growth model.) The heterogenous part describes the impact of factors exogenous to the demographic system, such as policy intervention.

The dynamics of multiregional population systems (the homogenous part of the equation) are governed by the interaction of fertility, mortality, and migration. In recent years demographers, geographers, economists, and planners have devoted their attention to modeling these dynamics in order to describe and explain the changes taking place in actual human-settlement systems. The models that have been developed have a similar underlying structure. In most instances they appear as a system of linear difference equations, or they may be transformed into such a system. The general format of the models is the matrix equation

$$x(t+1) = G(t)x(t), \quad (1)$$

where $x(t)$ is the vector⁽¹⁾ of population distribution at time t , and $G(t)$ is the population growth matrix at time t , which in most cases is assumed to be constant over time: $G(t) = G$. This model does not consider exogenous contributions to population growth—these will be added later.

Depending on the aggregation level, $x(t)$ is the population by region, or the population by age and region. Matrix models of aggregate multiregional population change are, for example, the Markov-chain model, the input-output model, and the components-of-change model. The model of disaggregate multiregional population change is known as the multiregional cohort-survival model (Rogers, 1975, chapter 5; see also Rees and Wilson, 1975).

(1) Note that all vectors are written as *column* vectors.

In the components-of-change model and the cohort-survival model, population at time t and its regional and/or age distribution depends only on the population distribution in the previous period. They are purely demographic models since they do not include other socioeconomic variables. In this closed system, the predetermined variables consist of lagged endogenous variables. The growth path of the system is completely determined by the growth matrix G and the initial conditions.

Both the components-of-change model and the multiregional cohort-survival model take the form of a system of homogenous first-order difference equations. They describe the dynamics of a *closed* multiregional system. The transformation of these models to open systems is straightforward. We add to system (1) a vector $e(t)$:

$$x(t+1) = G(t)x(t) + e(t), \quad (2)$$

which then describes the exogenous contributions to population growth, such as external migration.

To make the models more realistic, we extend the set of predetermined variables to include economic variables such as income, employment, housing stock, accessibility, and several types of government expenditures. Some of the predetermined variables are *controllable* by the policymaker, and are labeled *policy variables*, *control variables*, or *instrument variables*. Others are uncontrollable but are exogenously given. This classification of controllable and uncontrollable variables follows Tinbergen (1963). In the case of human-settlement policy, governments usually do not intervene directly in migration flows. Government control is mostly limited to socioeconomic instruments affecting migration. The modeling of such indirect population-distribution policies is discussed in Stern (1974), Wilson (1974), Cordey-Hayes (1975), MacKinnon (1975), and Willekens and Rogers (1977), among others.

The complete policy model may be expressed as ⁽²⁾

$$x(t+1) = G(t)x(t) + D(t)v(t) + L(t)u(t) + e(t), \quad (3)$$

where $v(t)$ is the vector of controllable variables, $u(t)$ is the vector of uncontrollable predetermined socioeconomic variables, and $D(t)$ and $L(t)$ are matrix multipliers. For simplicity, and without loss of generality, we delete the uncontrollable predetermined variables. Model (3) then reduces to

$$x(t+1) = G(t)x(t) + D(t)v(t). \quad (4)$$

The control vector $v(t)$ consists of socioeconomic instrument variables affecting the distribution of the population. The matrix multiplier $D(t)$ is important in this setting. An element $d_{ij}(t)$ denotes the impact on the population in region-age combination i of a unit change in the j th instrument at step t . In many cases the elements of this matrix are also assumed constant over time: $D(t) = D$. This implies that the effects of certain policies on the population distribution are independent of the time period when the policies are implemented. This is consistent with the Markovian assumption of time homogeneity. The linearity of equation (4) implies that the effects of the various policies are additive.

Equation (4) is the state equation of a state-space model. How it may be derived from linear demometric models describing the interdependence between demographic and socioeconomic variables is described in Willekens (1976b). The rationale for using state-space model (4) as the analytical or numerical tool for population-policy analysis, is that the homogenous part of equation (4) is exactly the demographic-growth model (components-of-change or cohort-survival) that describes the population

⁽²⁾ The fact that equation (3) is a first-order difference equation is by no means restrictive. Higher-order difference equations may be converted into a system of first-order difference equations (Zadeh and Desoer, 1963).

growth without intervention. The logical extension of population-growth models to policy models is therefore the addition of a heterogeneous part to the growth model (see also Rogers, 1966; 1968, chapter 6; 1971, pages 98–108). The resulting model is a heterogeneous system of linear first-order difference equations.

2.2 The constraints

Policymaking is subject to constraints. Relocating people or intervening in their residential-location decisions incurs a cost which may be political, economic, and/or social. The values that the control and state vectors $v(t)$ and $x(t)$ in equation (4) can take on are therefore restricted by political, economic, and social considerations. For example, let $v(t)$ denote the number of in-migrants, from outside the system, that have to move in in order to achieve certain population-distribution objectives. It is politically and socially unacceptable to relocate a very large number of people during a short time period. Therefore there is an upper bound to the number of in-migrants during a unit time period (Evtushenko and MacKinnon, 1976, page 639):

$$\sum_i v_i(t) \leq v(t), \quad t = 0, 1, \dots, T-1, \quad (5)$$

where the scalar $v(t)$ is the total in-migration pool available in the t th time period, and the $v_i(t)$ are the components of $v(t)$.

Instead of restricting the control vector by defining a total in-migration pool, each component of $v(t)$ may be required to lie within a lower and an upper bound:

$$\underline{v}_i(t) \leq v_i(t) \leq \bar{v}_i(t), \quad i = 1, 2, \dots, s. \quad (6)$$

Population-redistribution policy is not free. The imposition of controls implies the incurrence of costs. It is therefore natural to assume a budget constraint limiting the actions of the policymaker. We distinguish between a budget constraint for each period,

$$[c(t)]^T v(t) \leq C(t), \quad t = 0, 1, \dots, T-1, \quad (7)$$

and a global budget constraint,

$$\sum_{t=0}^{T-1} [c(t)]^T v(t) \leq C, \quad (8)$$

where a component $c_i(t)$ of the cost vector $c(t)$ denotes the cost of transferring a person to region i in the t th time period, where $C(t)$ is the total budget available during period t , and where C is the global budget.

Frequently the population distribution itself is constrained in addition to the control vector. For example, in a pure redistribution policy, the total population X of the system is held constant:

$$\sum_{j=1}^n x_j(t) = X = \sum_{j=1}^n x_j(0), \quad t = 1, 2, \dots, T. \quad (9)$$

As in the case of the control vector, the policymaker may want to put lower and upper bounds on the population in each region. This would avoid the excessive growth of some regions and the depopulation of others:

$$\underline{x}_j(t) \leq x_j(t) \leq \bar{x}_j(t), \quad t = 1, 2, \dots, T; \quad j = 1, 2, \dots, n. \quad (10)$$

A constraint receiving considerable attention in recent years is the resource constraint. Scarce resources include not only capital, but also raw materials, water, and the environment. As mentioned in the introduction to this paper, human-settlement policies in most countries are directed toward the conservation of those resources. This commitment must be reflected in the planning model. Therefore we

introduce the resource constraint⁽³⁾:

$$P(t)x(t) + S(t)v(t) \leq r(t), \quad t = 0, 1, \dots, T, \quad (11)$$

where $r(t)$ is the vector of available resources in the t th time period; the matrices $P(t)$ and $S(t)$ are rectangular matrices: an entry $p_{kj}(t)$, for example, denotes the amount of resource k required by an individual in region j during time period t ; an entry $s_{ki}(t)$ denotes the use of resource k per unit of control i during period t . Note that constraint (7) is a special case of constraint (11) in which a single resource, capital, is considered associated with the control.

Another constraint relates to the quality-of-life or income levels. Let $q(t)$ be the vector denoting the regional distribution of required quality-of-life levels. The quality-of-life constraint is then

$$M(t)x(t) + N(t)v(t) \geq q(t), \quad t = 0, 1, \dots, T, \quad (12)$$

where an entry $m_{lj}(t)$ of $M(t)$ denotes the per-capita level of the quality-of-life index l in region j at time t , and an entry $n_{li}(t)$ of $N(t)$ represents the impact of policy variable i on the level of the quality-of-life index l .

A final group of restrictions on the action span of the policymaker is represented by the boundary conditions. Since the planning of settlement systems starts from the current population distribution, we have the initial condition

$$x(0) = x^{\text{init}}. \quad (13)$$

On the other hand, the population distribution at the planning horizon, $x(T)$, may be fixed,

$$x(T) = x^{\text{hor}}, \quad (14)$$

or kept free.

2.3 The objective function

The objective function provides the explicit expression of the policymaker's preference system. The ultimate goal of national settlement-system planning is to increase the quality of life. There is no agreement on the factors determining the quality of life, and even less on its quantitative measurement. Therefore the usual difficulties in selecting the objective function arise. However, it should be noted that the optimization procedure cannot be considered as a final stage in the planning process, but only as a tool for analysing the connection between policy alternatives and system performance. Thus the analysis of 'optimal' solutions with different objective functions is needed. So, for example, the quality-of-life goal is replaced by a single objective involving monetary costs and benefits only. Such an objective function is given in equation (15). It is necessary to maximize the total benefit $B(v)$,

$$B(v) = \sum_{t=0}^T [\alpha(t)]^T x(t) + [\beta(t)]^T v(t), \quad (15)$$

where $\alpha(t)$ is the vector of unit benefit associated with the regional population levels at step t , and $\beta(t)$ is the vector of unit benefit associated with the controls. [If necessary, discounting is included in $\alpha(t)$ and $\beta(t)$.]

An objective function involving costs is shown in equation (16). The problem is to minimize

$$K(v) = \sum_{t=0}^T [\gamma(t)]^T x(t) + [\delta(t)]^T v(t), \quad (16)$$

⁽³⁾ Note that for two vectors y and z , $y \leq z$ means that $y_i \leq z_i$ for each pair of components y_i and z_i .

where $\gamma(t)$ is the vector of unit costs associated with the regional population levels at step t , and $\delta(t)$ is the vector of unit costs associated with the controls.

Another type of objective function is connected with optimization of the terminal state, and can be formulated in the following way:

$$B = \sum_{j=1}^n w_j(T)x_j(T) = [w(T)]^T x(T), \quad (17)$$

where $w_j(T)$ is the weighting coefficient of a population group $x_j(T)$.

In some instances, the policymaker may not want to minimize the costs associated with the settlement system and with the intervention in this system. Instead he may just want to bring the population distribution as close as possible to a desired target distribution $\bar{x}(T)$ at the planning horizon. This problem has been treated by Willekens (1976b, pages 66–85) for the two cases where explicit analytical solutions can be derived: the problems of initial-period control and linear-feedback control. In DLP problems the performance index may be formulated as

$$\text{minimize } \left\{ B(v) = \sum_j w_j(T) |x_j(T) - \bar{x}_j(T)| \right\}, \quad (18)$$

where $| \cdot |$ denotes absolute value and $w_j(T)$ is the weighting coefficient.

An interesting formulation of the objective at the end of the planning horizon is as follows:

$$B(v) = \min_{1 \leq j \leq n} \left\{ \frac{x_j(T)}{k_j} \right\}, \quad (19)$$

where the numbers k_j define the desired proportions of the terminal distribution.

Models with both types of nonlinear objective functions (18) and (19) can be easily reduced to linear-programming problems (Charnes and Cooper, 1961; Kantorovitch, 1965).

3 Dynamic linear-programming theory and methods

The purpose of this section is to describe DLP theory and methods in relation to models of national settlement-system planning. In the DLP framework, the dynamics of settlement systems are described by linear state equations; the feasible set of policies is delineated by linear constraints; and the preference system of the policymaker is described by a linear function. Thus we have a special type of linear-programming problem.

The impact of linear-programming models and methods on the practice of decisionmaking is well known (Dantzig, 1963; Kantorovitch, 1965). However, both LP theory itself and the basic range of its applications are of a one-stage, static nature. When the system to be optimized is changing over time, and its development is to be planned, a static approach is inadequate, and the problem of optimization becomes a dynamic multistage problem.

The principal feature of models of settlement planning is their dynamic character. On the other hand, the basic relations and conditions in such models often are linear. Hence DLP may be an efficient tool for elaborating optimal policies in large-scale, national settlement-system planning. For dynamic LP, new problems arise. For example, the basic question of the static LP problems is how to determine the optimal decision. In dynamic problems, however, knowledge of the sequence of optimal decisions is not sufficient. Of equal importance is the realization of these decisions (questions of feedback control, stability, and sensitivity analysis of optimal systems).

Our exposition of DLP theory and methods is divided into three sections. First, it is shown how demographic DLP models can be reduced to a canonical form, thus enabling the development of a unified approach for a whole range of problems of

national settlement planning which arise in practice. Second, there is an analysis of duality relations. Third, DLP computational methods are described and evaluated. The exposition is kept general and introductory. For more details on the several aspects of DLP theory and methods, the reader is referred to Propoi (1976; 1977).

3.1 The DLP canonical form

The models of multiregional population policy described earlier may be reduced to a canonical form. Before doing so, however, it is useful to recall the components of the DLP problem:

- the *state (development) equations* of the systems, with the distinct separation of state and control variables;
- the *constraints* imposed on these variables;
- the *planning period* (the number of stages during which the system is considered);
- and
- the *performance index (objective function)* which quantifies the quality of a control.

As we have noted earlier, state equations have the following form:

$$\mathbf{x}(t+1) = \mathbf{G}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{v}(t) + \mathbf{e}(t), \quad (20)$$

where the vector $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]$ defines the *state* of the system at stage t in the state space, the vector $\mathbf{u}(t) = [u_1(t), \dots, u_s(t)]$ defines the exogenous uncontrolled variables (known a priori in the deterministic models), $\mathbf{G}(t)$ is the $n \times n$ state-transformation matrix (in the majority of demographic problems $\mathbf{G}(t) = \mathbf{G}$ is the growth matrix), and $\mathbf{D}(t)$ is the $n \times s$ control-transformation matrix which defines the influence of a control on the state of the system.

Constraints imposed on the state and control variables may be written in general form as

$$\mathbf{P}(t)\mathbf{x}(t) + \mathbf{S}(t)\mathbf{v}(t) \leq \mathbf{r}(t) \quad \text{and} \quad \mathbf{v}(t) \geq 0, \quad (21)$$

where $\mathbf{r}(t) = [r_1(t), \dots, r_m(t)]$ is a given vector, and $\mathbf{P}(t)$ and $\mathbf{S}(t)$ are known $m \times n$ and $m \times s$ matrices.

The planning period T is assumed to be fixed. It is also assumed that the initial state of the system is given:

$$\mathbf{x}(0) = \mathbf{x}^{\text{init}}. \quad (22)$$

The objective function has the following general form:

$$B_1(\mathbf{v}) = [\alpha(T)]^T \mathbf{x}(T) + \sum_{t=0}^{T-1} \{ [\alpha(t)]^T \mathbf{x}(t) + [\beta(t)]^T \mathbf{v}(t) \}, \quad (23)$$

where $\alpha(t)$ ($t = 0, 1, \dots, T$) and $\beta(t)$ ($t = 0, 1, \dots, T-1$) are given weight coefficients.

The following definitions are used:

- the vector sequence $\mathbf{v} = \{\mathbf{v}(0), \dots, \mathbf{v}(T-1)\}$ is a *control (policy)* of the system;
- the vector sequence $\mathbf{x} = \{\mathbf{x}(0), \dots, \mathbf{x}(T)\}$, which corresponds to control $\mathbf{v}$ from state equations (20) with the initial state $\mathbf{x}(0)$, is the *system's trajectory*;
- the process $\{\mathbf{v}, \mathbf{x}\}$ is said to be *feasible* if it satisfies all the constraints of the problem [constraints (20)–(22) in this case];
- the feasible process $\{\mathbf{v}^*, \mathbf{x}^*\}$, maximizing the performance index (23), is known as the *optimal process*.

Hence the canonical form of the dynamic policy problem is formulated as follows.

Problem 1 (primal): Given the initial population distribution,

$$\mathbf{x}(0) = \mathbf{x}^{\text{init}}; \quad (24)$$

the state equations,

$$x(t+1) = G(t)x(t) + D(t)v(t) + e(t), \quad (25)$$

where

$x(t)$ is the population distribution at time t (state of the system),
 $v(t)$ is the vector of controls;
 $G(t)$ is the population-growth matrix (usually constant over time),
 $D(t)$ is the matrix expressing the impact of the control $v(t)$ on the population distribution $x(t)$, and
 $e(t)$ describes the exogenous contributions to population growth;
 and the constraints,

$$P(t)x(t) + S(t)v(t) \leq r(t) \quad \text{and} \quad v(t) \geq 0, \quad (26)$$

where

$r(t)$ is the vector of available resources at time t ,
 $P(t)$ is the matrix denoting the amount of resources required per individual at step t , and
 $S(t)$ is the matrix denoting the consumption of resources per unit of control at step t ;

find a control (policy) $v = \{v(0), \dots, v(T-1)\}$ and a corresponding state trajectory $x = \{x(0), \dots, x(T)\}$ that maximizes the performance index

$$B_1(v) = [\alpha(T)]^T x(T) + \sum_{t=0}^{T-1} \{[\alpha(t)]^T x(t) + [\beta(t)]^T v(t)\}, \quad (27)$$

where

$\alpha(t)$ is the vector of unit benefit associated with the regional population distribution $x(t)$, and
 $\beta(t)$ is the vector of unit benefit associated with the control $v(t)$.

The choice of a canonical form for problem 1 is to some extent arbitrary, various modifications and particular cases of problem 1 being possible. Some of them have been considered in the first part of this paper, and a classification of these modifications is given in the appendix. It should be noted that these modifications can be transformed one into the other. For example, let us consider the problems 1 and 1a, where problem 1a is the same as problem 1 except that it has performance index (17). Introducing a new variable $x_0(t)$ ($t = 0, \dots, T$), subject to

$$x_0(t+1) = x_0(t) + [\alpha(t)]^T x(t) + [\beta(t)]^T v(t) \quad \text{and} \quad x_0(0) = 0, \quad (28)$$

one can see that

$$x_0(T) = \sum_{t=0}^{T-1} \{[\alpha(t)]^T x(t) + [\beta(t)]^T v(t)\}. \quad (29)$$

Thus problem 1 will have the form of problem 1a with the performance index

$$B_1 = [\tilde{\alpha}(T)]^T \tilde{x}(T) \quad (30)$$

and the state equations

$$\tilde{x}(t+1) = \tilde{G}(t)\tilde{x}(t) + \tilde{D}(t)v(t) + e(t), \quad (31)$$

where

$$\tilde{\alpha}(T) = [1, \alpha_1(T), \dots, \alpha_n(T)];$$

$$\tilde{x}(0) = (0, x^{\text{init}}) \quad \text{and} \quad \tilde{x}(t) = [x_0(t), x_1(t), \dots, x_n(t)];$$

and where

$$\tilde{G}(t) = \begin{bmatrix} 1 & \alpha(t) \\ 0 & G(t) \end{bmatrix} \quad \text{and} \quad \tilde{D}(t) = \begin{bmatrix} 0 & \beta(t) \\ 0 & D(t) \end{bmatrix}$$

(where 0 is the zero vector of appropriate size).

Analogously, performance indices (18) and (19) can be reduced to (17). For example, performance index (19) can be replaced by the problem

$$\text{maximize}\{B(v) = \theta\} \quad (32)$$

with additional terminal state constraints

$$x_j(T) \leq \theta k_j, \quad j = 1, \dots, n. \quad (33)$$

If we consider problem 1a, with constraints (8), and introduce a variable $x_{n+1}(t)$, subject to the state equations

$$x_{n+1}(t+1) = x_{n+1}(t) + [c(t)]^T v(t), \quad t = 0, 1, \dots, T-1, \quad \text{and} \quad x_{n+1}(0) = 0, \quad (34)$$

then we obtain problem 1 with equations

$$\hat{x}(t+1) = \hat{G}(t)\hat{x}(t) + \hat{D}(t)v(t) + e(t), \quad (35)$$

where

$$\hat{x}(t) = [x_1(t), \dots, x_n(t), x_{n+1}(t)],$$

$$\hat{G}(t) = \begin{bmatrix} G(t) & 0 \\ 0 & I \end{bmatrix}, \quad \text{and} \quad \hat{D}(t) = \begin{bmatrix} D(t) & 0 \\ c(T) & I \end{bmatrix}$$

(where 0 and I are the appropriate zero and identity matrices respectively), and only one terminal condition is imposed:

$$\hat{x}(T) \leq \hat{C}, \quad (36)$$

where $\hat{C} = (0, \dots, 0, C)$.

These reasonings show that it is sufficient to develop solution methods only for problem 1 in order to obtain the solution methods for the whole set of DLP problems arising in case studies. But before discussing these methods let us consider some important theoretical properties of the DLP problems.

3.2 DLP theory

Problem 1 is an optimal-control problem with state equation (25), initial condition (24), constraints on state and control variables (26), and performance index (27). However, it may also be considered as a 'large' LP problem with constraints on variables in the form of equalities (24) and (25) and inequalities (26). In this case, problem 1 reduces to an LP problem with the staircase constraint matrix (table 1).

For the numerical solution of problem 1, one can therefore rely on a standard LP computer code. However, this straightforward approach to solving DLP problems is inefficient for two reasons. First, the 'static' LP model thus arrived at is so large in real cases that it cannot be solved effectively even by using the most up-to-date computers. Second, and more important, even if the optimal solutions of DLP problem 1 are found by conventional means, problems of implementing this solution still exist. These reasons provide the rationale for the development of DLP theory and methods.

Duality theory plays a key role in optimization methods. It permits the replacement of the original primal problem by some equivalent dual. It should be stressed that this equivalent dual can be interpreted in real terms for all real cases,

thus enabling one to understand more deeply the original problem. The following results are true for DLP problem 1 (Propoi, 1977).

Problem 2 (dual): Find the *dual control* $\lambda = \{\lambda(T-1), \dots, \lambda(1), \lambda(0)\}$ and the associated *dual trajectory* $\pi = \{\pi(T), \dots, \pi(1), \pi(0)\}$ satisfying the *costate (dual) equation*

$$\pi(t) = [G(t)]^T \pi(t+1) - [P(t)]^T \lambda(t) + \alpha(t) \quad (37)$$

with the boundary condition

$$\pi(T) = \alpha(T), \quad (38)$$

subject to the constraints

$$[D(t)]^T \pi(t+1) - [S(t)]^T \lambda(t) \leq -\beta(t) \quad \text{and} \quad \lambda(t) \geq 0, \quad (39)$$

and minimizing the objective function

$$B_2(\lambda) = [\pi(0)]^T x^{\text{init}} + \sum_{t=0}^{T-1} \{[p(t+1)]^T e(t) - [r(t)]^T \lambda(t)\}, \quad (40)$$

where $\pi(t) = [\pi_1(t), \dots, \pi_n(t)]$ and $\lambda(t) = [\lambda_1(t), \dots, \lambda_m(t)]$, for $\lambda_k(t) \geq 0$, $k = 1, \dots, m$, are Lagrange multipliers for constraints (24), (25), and (26) respectively.

Dual problem 2 is also a control-type problem, as is Primal problem 1. Here the variable $\lambda(t)$ is a dual control and $\pi(t)$ is a dual or a costate of the system. The costate equations describe the dual system. Note that we have reversed time in dual problem 2: $t = T-1, \dots, 1, 0$.

The dual state and control variables (shadow prices) provide valuable information on marginal costs and marginal values for problem 1. They enable one to analyse the effects of changes in the coefficients of the objective function, constraints, growth matrix, etc.

For dual problems 1 and 2, a number of duality relations hold. They are stated here as theorems without proofs—for details see Propoi (1977).

Theorem 1 (the DLP 'global' duality conditions): (1) For any feasible primal control v and dual control λ , the inequality

$$B_1(v) \leq B_2(\lambda) \quad (41)$$

holds. (2) The solvability of either problem 1 or problem 2 implies the solvability of the other, with

$$B_1(v^*) = B_2(\lambda^*), \quad (42)$$

Table 1. Staircase control structure.

Variable									Inequality or equality of constraints	Right-hand side constant
$v(0)$	$x(1)$	$\dots$	$x(t)$	$v(t)$	$x(t+1)$	$\dots$	$x(T-1)$	$v(T-1)$		
$-D(0) \quad I$									$=$	$e(0) + G(0)x^{\text{init}}$
$S(0)$									$\leq$	$r(0) + P(0)x^{\text{init}}$
$\vdots$									$\vdots$	$\vdots$
$-G(t) \quad -D(t) \quad I$									$=$	$e(t)$
$P(t) \quad S(t)$									$\leq$	$r(t)$
$\vdots$									$\vdots$	$\vdots$
$-G(T-1) \quad -D(T-1) \quad I$									$=$	$e(T-1)$
$P(T-1) \quad S(T-1)$									$\leq$	$r(T-1)$
Performance-index constant										
$\beta(0)$	$\alpha(0)$	$\dots$	$\alpha(t)$	$\beta(t)$	$\alpha(t+1)$	$\dots$	$\alpha(T-1)$	$\beta(T-1)$	$\alpha(T)$	maximize

where v^* and λ^* are optimal controls (that is, with identical values of the performance index for problems 1 and 2).

Equality (42) shows that the solution of primal problem 1 can be replaced by the solution of dual problem 2, and inequality (41) gives the upper bound of the value of the performance index in problem 1.

The duality relations on theorem 1 are 'global' in character, and are stated for the whole planning period. The duality relations can also be formulated in a local decomposable way for each time period t , $t = 0, 1, \dots, T-1$. For this purpose, let us introduce the Hamiltonian

$$H_1[\pi(t+1), v(t)] = [\beta(t)]^T v(t) + [\pi(t+1)]^T D(t)v(t) \quad (43)$$

for primal problem 1 and

$$H_2[x(t), \lambda(t)] = [\lambda(t)]^T r(t) - [\lambda(t)]^T P(t)x(t) \quad (44)$$

for dual problem 2.

Theorem 2 (the DLP 'local' duality conditions): (1) For any feasible processes $\{v, x\}$ and $\{\lambda, \pi\}$, the following inequalities hold:

$$H_1[\pi(t+1), v(t)] \leq H_2[x(t), \lambda(t)], \quad t = 0, \dots, T-1. \quad (45)$$

(2) For any feasible process $\{v^*, x^*\}$ of the primal and $\{\lambda^*, \pi^*\}$ of the dual to be optimal, it is necessary and sufficient that the values of Hamiltonians are equal:

$$H_1[\pi^*(t+1), v^*(t)] = H_2[x^*(t), \lambda^*(t)], \quad t = 0, \dots, T-1. \quad (46)$$

Theorem 2 shows that in order to investigate a pair of dual dynamic problems such as problems 1 and 2 it is sufficient to consider the following pair of dual 'local' (static) problems of LP:

$$\left. \begin{array}{l} \text{maximise}\{H_1[\pi(t+1), v(t)]\} \\ \text{subject to} \\ P(t)x(t) + S(t)v(t) \leq r(t) \\ \text{and } v(t) \geq 0, \\ \text{for } t = 0, \dots, T-1, \end{array} \right\} \quad (47)$$

and

$$\left. \begin{array}{l} \text{minimise}\{H_2[x(t), \lambda(t)]\} \\ \text{subject to} \\ [D(t)]^T \pi(t+1) - [S(t)]^T \lambda(t) \leq -\beta(t) \\ \text{and } \lambda(t) \geq 0, \\ \text{for } t = T-1, \dots, 1, 0. \end{array} \right\} \quad (48)$$

Any of the 'static' duality relations of LP optimality conditions (Dantzig, 1963) for the pair of dual LP problems (47) and (48), linked by the state equations (24) and (25), and (37) and (38), determine the corresponding optimality conditions for the pair of dual DLP problems 1 and 2. Such conditions have been formulated above; in a similar manner the following important optimality conditions are obtained (Propoi, 1973; 1977):

Theorem 3 (maximum principle for primary problem 1): For a control v^* to be optimal in primary problem 1, it is necessary and sufficient that there exists a feasible process $\{\lambda^*, \pi^*\}$ of dual problem 2 such that, for $t = 0, 1, \dots, T-1$, the equality

$$\max\{H_1[\pi^*(t+1), v(t)]\} = H_1[\pi^*(t+1), v^*(t)]$$

holds, where the maximum is taken over all $v(t)$ satisfying constraints (26), and $\lambda^*(t)$ is the optimal dual variable in LP problem (48).

The minimum principle can be formulated in a similar way for dual problem 2 (Propoi, 1977).

The Hamiltonian functions may be interpreted as 'local' objective functions for each time period t , which consist (for primal problems) of the current profit, $[\alpha(t)]^T x(t) + [\beta(t)]^T v(t)$, modified by the shadow price vector, $\pi(t+1)$ —that is, with $[\pi(t+1)]^T D(t)v(t)$ instead of $[\alpha(t)]^T x(t)$ —accounting for the dynamics of the process. The maximum principle for problem 1 states that if we know the optimal values of shadow prices $\pi^*(t+1)$ [or $\lambda^*(t)$] then it is possible, for each current state $x(t)$, to solve only the 'local' static LP problem (47).

3.3 DLP computational methods

Simple DLP problems can be handled by standard LP codes. DLP problems of a realistic size require, however, the development of special DLP methods. We shall distinguish between finite and iterative methods.

DLP *finite methods* find an optimal solution in a finite number of steps and are a further development of large-scale LP methods to dynamic problems. First, we mention the extension of the well-known simplex method to DLP problems (Propoi and Krivonozhko, 1977). The dynamic simplex method is used to obtain exact optimal solutions of DLP problems, for a finite number of steps, by treating at each step only the set of T local bases of dimension $m \times m$ [m is the number of constraint rows in inequalities (21)] instead of working with the global basis of dimension $mT \times mT$. The method is closely related to the most effective large-scale LP methods based on factorization of the constraint matrix. The second group of finite methods is based on decomposition methods of LP, especially on the Dantzig-Wolfe decomposition principle. This technique was applied to DLP problems by Glassey (1970), Ho and Manne (1974), and Krivonozhko (1977).

Iterative methods do not produce exact solutions in a finite number of iterations. But in many cases the approximation is quite adequate. In addition, the iterative methods are characterized by simplicity of computer coding, low demands on computer memory, and low sensitivity to the disturbances. A DLP iterative method is considered in Propoi and Yadykin (1975; 1976).

Unlike for static LP, the realization of the optimal solution in dynamic problems is as important as its determination. Such questions as the realization of the optimal solution as a program—that is, depending on the number of states: $v^*(t)$, $t = 0, \dots, T-1$ —or as a feedback control—that is, depending on the current value of states: $v^*(t) = v_t^*[x(t)]$, $t = 0, \dots, T-1$; the stability and sensitivity of the optimal system; and the connection of optimal solutions for long- and short-range models must be answered.

Naturally all the practical problems of national settlement planning cannot be kept within the format of DLP. In some cases the objective function is stated as a quadratic or nonlinear (convex) function of state and control variables (Willekens, 1976b). The extension of DLP methods to quadratic and convex dynamic-programming problems can be developed in a way similar to the static methods (see Dantzig, 1963).

When the exogenous variables cannot be given a priori, we come to DLP problems with uncertainty conditions. They can be formalized using stochastic optimization methods (Ermoljev, 1975) or max-min methods (Propoi and Yadykin, 1974a; 1974b). Such optimization techniques may be conceptually superior in some instances, but operational problems are more formidable because of data requirements and solution difficulties.

4 Conclusion

In this introductory paper, we sketched the basic idea of applying DLP to the planning of national settlement systems. The DLP approach requires a linear model. Several situations may arise in which these conditions are satisfied. For instance, most currently available multiregional population-growth models are linear. Moreover a wide variety of DLP formulations are possible. It has been shown in this paper how they may be reduced to the standard or canonical form.

Interest in dynamic linear programming started recently and the theory and algorithms are still being developed. DLP, which combines linear-programming and optimal-control theory, provides a tool for the solution of large-scale dynamic problems. The second section of this paper has given a short overview of recent findings in DLP theory and of algorithms which are relevant for formulating problems of national settlement-system planning. We did not intend to present an in-depth analysis of the theory. Our aim was merely to show the potential of DLP to the planning of human-settlement systems. Considerable research is still needed, both theoretical and applied. Interesting topics of further theoretical investigation are, for example, the interpretation of the basic dual relations in demographic terms or the investigation of linkages between demographic and economic models.

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APPENDIX

Alternative formulations of the DLP problem

The purpose of this appendix is to present some modifications and variants of problem 1. For example, *state equations* may include matrices and/or vectors that are independent of the stage t —equations S2. The external disturbance $e(t)$ may vanish—equations (4). Equations S3, for example, are obtained from the difference approximation of the continuous analog of problem 1.

An important class of DLP problems is the system with delays in state and/or control variables—equation S4. For example, in a demographic system the state $x(t+1)$ at the step $t+1$ may depend on certain previous states, $x(t-n_1)$, $x(t-n_2)$, ..., $x(t-n_\nu)$, and on certain previous control actions, $v(t-s_1)$, $v(t-s_2)$, ..., $v(t-s_\mu)$.

Constraints on the state control variables can be in the form of equalities C2 [as in equations (9)] or in the form of separate inequalities C3 and C4 [as in inequalities (5)–(7)]. These variables can have additional restrictions on sign—inequalities C5 and C6. In some cases the constraints should be considered in the summarized form of inequalities C7 or C8 [as in inequalities (8)].

It is useful to single out the constraints on the left- and/or right-hand side of the trajectory (boundary conditions). For example, the left- and/or right-hand side of the trajectory can be fixed—B1 and B3—or free—B2 and B4.

The number of steps T of the *planning period* can be fixed—P1—or may be defined by some conditions on the terminal state—for example, B3. A typical problem here is to bring a demographic system to a desired population distribution in a minimal number of steps T .

The value of the *objective function* can depend only on the trajectory $\{x(t)\}$ or on the control sequence $\{v(t)\}$, or may be determined only by the terminal state $x(T)$ of the trajectory—equation I2 [as in equations (17)–(19)].

A number of variants of problem 1 are now illustrated.

State equations

$$S1 \quad x(t+1) = G(t)x(t) + D(t)v(t) + e(t).$$

$$S2 \quad x(t+1) = Gx(t) + Dv(t) + e.$$

$$S3 \quad x(t+1) = x(t) + G(t)x(t) + D(t)v(t) + e(t).$$

$$S4 \quad x(t+1) = \sum_{j=1}^{\nu} G(t-n_j)x(t-n_j) + \sum_{i=1}^{\mu} D(t-s_i)v(t-s_i).$$

Constraints

$$C1 \quad P(t)x(t) + S(t)v(t) \leq r(t).$$

$$C2 \quad P(t)x(t) + S(t)v(t) = r(t).$$

$$C3 \quad P(t)x(t) \leq r^{(1)}(t).$$

$$C4 \quad S(t)v(t) \leq r^{(2)}(t).$$

$$C5 \quad x(t) \geq 0.$$

$$C6 \quad v(t) \geq 0.$$

$$C7 \quad \sum_{\tau=0}^{t-1} [P(\tau)x(\tau) + S(\tau)v(\tau)] \leq r(t), \quad t = 1, \dots, T.$$

$$C8 \quad \sum_{t=0}^{T-1} [P(t)x(t) + S(t)v(t)] \leq r.$$

Boundary conditions

$$B1 \quad x(0) = x^{\text{init}}.$$

$$B2 \quad x(0) \text{ is free.}$$

$$B3 \quad x(T) = x^{\text{hor}}.$$

$$B4 \quad x(T) \text{ is free.}$$

*Planning period*P1 T is fixed.P2 T is free.*Performance indices*

$$\text{I1} \quad B_1(v) = [\alpha(T)]^T x(T) + \sum_{t=0}^{T-1} \{[\alpha(t)]^T x(t) + [\beta(t)]^T v(t)\}.$$

$$\text{I2} \quad B_1(v) = [\alpha(T)]^T x(T).$$