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Elementary inductive definitions in **HA**: from strictly positive towards monotone

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ABSTRACT

A study of elementary inductive definitions (e.i.d.) in **HA**. Strictly positive e.i.d. have closure ordinals $\leq \omega$, and define predicates that are already definable in **HA**. We enlarge this class by adding so-called *J*-operators, for example $\neg \neg$. E.i.d. in this larger class have closure ordinals up to $\omega + \omega$, but they are conservative over **HA** w.r.t. definability.

1. INTRODUCTION

We shall consider as inductive definitions formulae in the language of **HA** expanded with a single one place predicate variable P , containing at most one numerical variable free. The meaning of such an inductive definition $A(P, x)$ is the least fixed-point of $A(P, x)$, i.e. a predicate P^A satisfying

- (i): $\forall x(A(P^A, x) \leftrightarrow P^A x)$
- (ii): $\forall x(A(Q, x) \rightarrow Qx) \rightarrow \forall x(P^A x \rightarrow Qx)$.

So the inductive definition specifies the closure conditions of the predicate it defines. The question is: for which $A(P, x)$ can we justify the existence of such a P^A ? If $A(P, x)$ is *monotone*, i.e.

$$\forall x(Qx \rightarrow Rx) \rightarrow \forall x(A(Q, x) \rightarrow A(R, x)),$$

then we can approximate P^A from below; define

$$P_0^A x : \Leftrightarrow A(\lambda x \cdot \perp, x)$$

$$P_{\beta+1}^A x : \Leftrightarrow A(P_\beta^A, x)$$

$$P_\lambda^A x : \Leftrightarrow \exists \mu < \lambda P_\mu^A x, \text{ lim } \lambda$$

$$P_\infty^A x : \Leftrightarrow \exists \mu P_\mu^A x$$

Note that for monotone $A(P, x)$ (i) \leftarrow is redundant: we have $A(P^A, x) \rightarrow P^A x$ by (i) \rightarrow , then by monotonicity we get $A(A(P^A, \cdot), x) \rightarrow A(P^A, x)$, and finally by (ii) $P^A x \rightarrow A(P^A, x)$.

Classically P^A exists and is equal to the least fixed-point of $A(P, x)$. An *elementary* inductive definition (e.i.d.) is an inductive definition without an unbounded universal quantifier occurring in front of a positive subformula containing P , and without an unbounded existential quantifier in front of a negative subformula containing P ; the inductive definition must be monotone. Classically we know that for e.i.d. the approximation closes up at or before stage ω , so $P_\infty^A = P_\omega^A$. Intuitionistically, this is only true (in general) for strictly positive inductive definitions, i.e. formulae $A(P, x)$ built up from atomic formulae Pt , from **HA**-formulae φ (these do not contain P), by means of $\exists, \forall y < s, \wedge, \vee$.

Now we want to solve the following problems

- (i): give neat ordinal bounds for arbitrary e.i.d., not only for the strictly positive ones
- (ii): prove or refute: e.i.d. enhance the expressive power of **HA**.

I have no complete answer to these questions. I will describe special extensions of the class of strictly positive e.i.d., which do not enhance the expressive power of **HA**, while those e.i.d. may have a closure ordinal up to $\omega + \omega$. Those extensions are made by closing the strictly positive formulae under new operations, like $\neg \neg$. When we allow arbitrary monotone formulae, these problems look rather intractable. In particular, implication (with negative antecedent and positive consequent) seems rather tough.

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This article is a partial answer to a question, posed by Kreisel in [Kre63, p. 3.25]. I am indebted to prof. A.S. Troelstra for remembering it, and pointing it out to me.

CONVENTION

Throughout this article the symbols \leftrightarrow resp. \rightarrow and \Leftrightarrow resp. \Rightarrow stand for *provable* equivalence resp. consequence in a formal system. But only \leftrightarrow and \rightarrow are used as connectives in a formal language, while \Leftrightarrow and \Rightarrow denote equivalence resp. consequence relations between formulae.

2. EXAMPLES

2.1. CLOSURE AT $\omega + 1$

An e.i.d. that closes up at stage $\omega + 1$ (exactly). Let C be a nonrecursive RE-set, say

$$x \in C \leftrightarrow \exists z \text{Texz}; \text{ assume } \text{Texz} \rightarrow x \leq z.$$

Define, assuming that pairing is surjective:

$$A(P, \langle x, z \rangle) : \Leftrightarrow \exists m \leq z \text{ Texm} \vee P \langle x, z+1 \rangle.$$

Then

$$P_0^A \langle x, z \rangle \Leftrightarrow \exists m \leq z \text{ Texm}$$

$$P_1^A \langle x, z \rangle \Leftrightarrow \exists m \leq z \text{ Texm} \vee P_0^A \langle x, z+1 \rangle \Leftrightarrow \exists m \leq z+1 \text{ Texm}$$

⋮

$$P_k^A \langle x, z \rangle \Leftrightarrow \exists m \leq z+k \text{ Texm}$$

⋮

$$P_\omega^A \langle x, z \rangle \Leftrightarrow \exists m \text{ Texm} \Leftrightarrow x \in C.$$

We see quickly that $P_\omega^A = P_{\omega+1}^A$ and $P_k^A \neq P_\omega^A$. The last inequality follows from the fact that C is infinite and $\text{Texz} \rightarrow x \leq z$. Now we define, following [Kre63, pp. 3.6 and 3.24]:

$$B(P, x) : \Leftrightarrow A(P, x) \vee \neg \neg Px.$$

Then, for all $n < \omega$, $P_n^B x \leftrightarrow P_n^A x$, and P_n^A is recursive.

PROOF.

$$P_0^B x \Leftrightarrow P_0^A x \vee \neg \neg \perp \Leftrightarrow P_0^A x \text{ and clearly } P_0^A \text{ is recursive.}$$

$$P_{n+1}^B x \Leftrightarrow A(P_n^B, x) \vee \neg \neg P_n^B x \Leftrightarrow \text{ind hyp}$$

$$A(P_n^A, x) \vee \neg \neg P_n^A x \Leftrightarrow \text{def, ind hyp}$$

$$P_{n+1}^A x \vee P_n^A x \Leftrightarrow P_{n+1}^A x, \text{ and } P_{n+1}^A \text{ is recursive.}$$

□

Consider now P_ω^B , $P_{\omega+1}^B$ and $P_{\omega+2}^B$:

$$P_\omega^B x \Leftrightarrow \exists n P_n^B x \Leftrightarrow \exists n P_n^A x \Leftrightarrow P_\omega^A x.$$

$$P_{\omega+1}^B x \Leftrightarrow B(P_\omega^B, x) \Leftrightarrow A(P_\omega^A, x) \vee \neg \neg P_\omega^A x$$

$$\Leftrightarrow P_\omega^A x \vee \neg \neg P_\omega^A x \Leftrightarrow \neg \neg P_\omega^A x \not\equiv P_\omega^A x, \text{ for } P_\omega^A \text{ is nonrecursive.}$$

$$P_{\omega+2}^B x \Leftrightarrow A(P_{\omega+1}^B, x) \vee \neg \neg P_{\omega+1}^B x \Leftrightarrow A(\neg \neg P_\omega^A, x) \Leftrightarrow \neg \neg \neg \neg P_\omega^A x$$

$$\Leftrightarrow \neg \neg P_\omega^A x \text{ because } A(\neg \neg P_\omega^A, x) \Leftrightarrow \neg \neg A(P_\omega^A, x) \Leftrightarrow \neg \neg P_\omega^A x.$$

It is possible to construe e.i.d. $C(P, x)$ that close up at stage $\omega + \omega$, by exploiting this trick.

□ (first example)

2.2. CLOSURE AT $\omega + \omega$

We give an e.i.d. with closure ordinal $\omega + \omega$. Let $\langle \dots \rangle$ be a coding of sequences of natural numbers. Let $A(P, x)$ be an e.i.d. that defines a nonrecursive $P^A = P_\omega^A$, while the P_k^A are recursive (cf. the first example); in addition, let $P^A \subseteq \{\langle x \rangle \mid x \in \mathbb{N}\}$, and let $A(P, x)$ be insensitive to numbers outside this set, i.e.

$$A(P, x) \leftrightarrow A(\lambda y \cdot Py \wedge \exists z(\langle z \rangle = y), x).$$

Define

$$B(P, x) := (A(P, x) \wedge \text{lh } x = 1) \vee \\ \vee \exists y \exists z (Py \wedge \neg \neg A(P, z) \wedge \text{lh } z = 1 \wedge x = y * z).$$

Then $P^B = P_{\omega+\omega}^B$, by the following lemmas, whose proofs are not particularly interesting and not too difficult. Sometimes I use set-theoretic notation like $x \in P_\omega^A$ for $P_\omega^A x$.

LEMMA 2.1. $P_\omega^B = \{\langle x_1, \dots, x_k \rangle \mid k \in \mathbb{N}, \langle x_i \rangle \in P_\omega^A, i = 1, \dots, k\}$.

LEMMA 2.2.

$$P_{\omega+n}^B = \{\langle x_1, \dots, x_k \rangle \mid k > 0 \wedge \langle x_1 \rangle \in P_\omega^A \\ \wedge \forall i \in \{1, \dots, k - n\} \langle x_i \rangle \in P_\omega^A \\ \wedge \forall i \in \{k - (n + 1), \dots, k\} \langle x_i \rangle \in \neg \neg P_\omega^A\}.$$

LEMMA 2.3. $x \in P_{\omega+n+1}^B \not\leftrightarrow x \in P_{\omega+n}^B$.

LEMMA 2.4.

$$P_{\omega+\omega}^B = \bigcup_{n \in \omega} P_{\omega+n}^B = \\ = \{\langle x_1, \dots, x_k \rangle \mid k > 0 \wedge \langle x_1 \rangle \in P_\omega^A \wedge \langle x_2 \rangle, \dots, \langle x_k \rangle \in \neg \neg P_\omega^A\}.$$

It is clear from this construction, that the closure ordinal of B cannot be proved to be less than $\omega + \omega$.

3. J-OPERATORS

The following definition is meant as a generalization of the $\neg \neg$ -operator (cf. [FS73, pp. 324–334]):

DEFINITION 3.1. *A J-operator is an operator $J(\cdot)$, on **HA**-formulae, that is **HA**-definable, and that satisfies:*

- (i): $Q \rightarrow J(Q)$ (increasing)
 - (ii): $J(Q \wedge R) \leftrightarrow J(Q) \wedge J(R)$ (\wedge -distributive)
 - (iii): $J(J(Q)) \rightarrow J(Q)$ (idempotent)
- Note that from (ii)(\rightarrow) follows:*
- (iv): $(Q \rightarrow R) \rightarrow (J(Q) \rightarrow J(R))$ (monotone).

We do not allow J to have free variables.

DEFINITION 3.2. $P[P]$ is the class of strictly positive formulae, i.e.:

- Pt, t a term, is a formula of $P[P]$
- a formula φ of the language of **HA** is a formula of $P[P]$
- $P[P]$ is closed under $\exists, \forall^<, \wedge, \vee$.

$P(J)[P]$, J a J -operator, is defined analogously, except that $P(J)[P]$ is also closed under J .

FACT 3.1. For $A(P, x) \in P[P, x]$, $P^A = P_\omega^A$ is **HA**-definable. See [TvD88, Vol I, pp. 145–152].

THEOREM 3.2. For $A(P, x) \in P(J)[P, x]$, $P^A = P_{\omega+}^A$ is **HA**-definable.

Before giving the proof, I will supply some technical lemmas and hint at the idea behind the proof.

LEMMA 3.3. (Shifting J to the outside)

- (i): $J(P) \vee J(Q) \rightarrow J(P \vee Q)$
- (ii): $J(P) \wedge J(Q) \rightarrow J(P \wedge Q)$
- (iii): $\exists x J(A(x)) \rightarrow J(\exists x A(x))$
- (iv): $\forall x < t J(A(x)) \rightarrow J(\forall x < t A(x))$.

PROOF

- (i): $\left. \begin{array}{l} P \rightarrow P \vee Q \\ Q \rightarrow P \vee Q \end{array} \right\} \xrightarrow{\text{monotonicity}} \left. \begin{array}{l} J(P) \rightarrow J(P \vee Q) \\ J(Q) \rightarrow J(P \vee Q) \end{array} \right\} \Rightarrow J(P) \vee J(Q) \rightarrow J(P \vee Q)$
- (ii): by \wedge -distributivity (\leftarrow)
- (iii): $A(x) \rightarrow \exists x A(x) \Rightarrow J(A(x)) \rightarrow J(\exists x A(x)) \Rightarrow \exists x J(A(x)) \rightarrow J(\exists x A(x))$
- (iv): let J -SHIFT(y) denote the following schema:

$$\forall x(x < y \rightarrow J(A(x))) \rightarrow J(\forall x(x < y \rightarrow A(x))), y \notin FV(A).$$

We prove $\forall y J$ -SHIFT(y) by induction:

$$\forall x(x < 0 \rightarrow A(x)), \text{ so by increase: } J(\forall x(x < 0 \rightarrow A(x))).$$

$$\forall x(x < Sy \rightarrow J(A(x))) \quad \Rightarrow \text{“HA”}$$

$$\forall x(x < y \rightarrow J(A(x))) \wedge J(A(y)) \quad \Rightarrow \text{ind hyp}$$

$$J(\forall x(x < y \rightarrow A(x))) \wedge J(A(y)) \quad \Rightarrow \wedge\text{-distributivity}$$

$$J(\forall x(x < y \rightarrow A(x)) \wedge J(A(y))) \quad \Rightarrow \text{“HA under } J\text{”}$$

$$J(\forall x(x < Sy \rightarrow A(x))).$$

We conclude: for any term t :

$$\forall x(x < t \rightarrow J(A(x))) \rightarrow J(\forall x(x < t \rightarrow A(x))).$$

□ (lemma 3.3)

The comment “**HA**” means: by reasoning in **HA**; “**HA** under J ” means: by reasoning in **HA** in the scope of J ; this is justified by the fact that J is increasing and monotone.

DEFINITION 3.3. *Let $A(P)$ be a $P(J)[P]$ -formula. Occurrences of subformulae, used in the construction of $A(P)$, according to the definition of $P(J)[P]$, are called components.*

Remark that a $P(J)[P]$ -formula is monotone in its components, because $\exists, \forall^<, \wedge, \vee, J$ are all monotone connectives.

LEMMA 3.4. *Let $A(P)$ be a $P(J)[P]$ -formula. Let C be a component of $A(P)$ of the form $J(B(P))$, with at least one occurrence of P . Let $A'(P)$ be obtained from A by replacing that component $J(B(P))$ by $B(P)$. Then $A(P) \rightarrow J(A'(P))$. I.e.*

$$A(P) \equiv \dots J(B(P)) \dots$$

$$J(A'(P)) \equiv J(\dots B(P) \dots).$$

PROOF. Easy, by induction on the structure of $A(P)$. In fact, this is nothing else than repeatedly shifting J outwards, using the fact that a component occurs only in scopes of $\wedge, \vee, \exists, \forall^<, J$, and applying lemma 3.3.

□

4. DECOMPOSITION OF THE APPROXIMATION PROCESS

DEFINITION 4.1. *Let $A(P, x)$ be a $P(J)[P]$ -formula.*

\bar{A} : $\equiv A$ where every J with P in its scope has been deleted;

A^* : $\equiv A$ where every occurrence of P in the scope of J has been replaced by $P_\omega^{\bar{A}}$; so:

$$A(P) \equiv \dots P s_i \dots J(\dots P t_j \dots)$$

$$\bar{A}(P) \equiv \dots P s_i \dots \dots P t_j \dots$$

$$A^*(P) \equiv \dots P s_i \dots J(\dots P_\omega^{\bar{A}} t_j \dots).$$

REMARK

$\bar{A} \in P[P, x]$, so $P^{\bar{A}} = P_\omega^{\bar{A}}$ is **HA**-definable by the fact above; it follows that A^* is a $P[P, x]$ -formula, so $P^{A^*} = P_\omega^{A^*}$ is **HA**-definable too.

The idea of the proof is emerging: instead of iterating $A(P, x)$ indefinitely, we split the process in iterations that continue at most till stage ω . In the first iteration we neglect the J -operator completely, then we administer its effect one time; the second iteration also goes on without J -operator. The reason that this suffices, is mainly the idempotency of the J -operator.

LEMMA 4.1. *Let $A(P, x) \in P(J)[P, x]$. Then*

- (i): $P_\alpha^A x \rightarrow P_\alpha^A x$
- (ii): $J(P_\alpha^A x) \rightarrow J(P_\alpha^{\bar{A}} x)$.

PROOF. (i) follows from $\bar{A} \rightarrow A$, (ii) from $J(A) \rightarrow J(\bar{A})$, both by induction on α . Ad (i): A is obtained from \bar{A} by replacing components B by $J(B)$. Use increase ($B \rightarrow J(B)$) and monotonicity in components. Ad (ii): this is seen as follows: by repeatedly applying lemma 3.4 we have $A \rightarrow J(\bar{A})$; then, by monotonicity $J(A) \rightarrow J(J(\bar{A}))$ and by idempotency $J(A) \rightarrow J(\bar{A})$. Let us now carry out the induction for (ii):

$$\alpha = 0 \quad : \quad J(P_0^A x) \stackrel{\text{by def}}{\equiv} J(A(\lambda x \cdot \perp, x)) \Rightarrow \text{for } J(A) \rightarrow J(\bar{A}), \text{ see above}$$

$$J(\bar{A}(\lambda x \cdot \perp, x)) \stackrel{\text{by def}}{\equiv} J(P_0^{\bar{A}} x).$$

$$\alpha = \beta + 1 \quad : \quad J(P_{\beta+1}^A x) \stackrel{\text{by def}}{\equiv} J(A(P_\beta^A, x)) \Rightarrow \text{for } J(A) \rightarrow J(\bar{A}), \text{ see above}$$

$$J(\bar{A}(P_\beta^A, x)) \Rightarrow \bar{A} \text{ monotone, } J \text{ increasing}$$

$$J(\bar{A}(J(P_\beta^A), x)) \Rightarrow \text{ind hyp}$$

$$J(\bar{A}(J(P_\beta^{\bar{A}}), x)) \Rightarrow \text{lemma 3.4}$$

$$J(J(\bar{A}(P_\beta^{\bar{A}}), x)) \Rightarrow \text{idempotency}$$

$$J(P_{\beta+1}^{\bar{A}} x).$$

$$\text{lim } \alpha \quad : \quad J(P_\alpha^A x) \stackrel{\text{by def}}{\equiv} J(\exists \beta < \alpha P_\beta^A x) \Rightarrow J \text{ increasing}$$

$$J(\exists \beta < \alpha J(P_\beta^A x)) \Rightarrow \text{ind hyp, monotonicity of } J$$

$$J(\exists \beta < \alpha J(P_\beta^{\bar{A}} x)) \Rightarrow \text{lemma 3.4}$$

$$J(J(\exists \beta < \alpha P_\beta^{\bar{A}} x)) \Rightarrow \text{idempotency}$$

$$J(P_\alpha^{\bar{A}} x).$$

□

LEMMA 4.2. *Let $A(P, x) \in P(J)[P, x]$. Then*

- (i): $P_\infty^A x \leftrightarrow P_\omega^{A^*} x$
- (ii): $P_\omega^{A^*} x \leftrightarrow P_{\omega+\omega}^A x$.

PROOF. (i)(\rightarrow): by induction on α we prove $P_\alpha^A x \rightarrow P_\omega^{A^*} x$.

$$\alpha = 0 \quad : \quad P_0^A x \Leftrightarrow A(\lambda x \cdot \perp, x) \Rightarrow P_0^{A^*} x \text{ (since } \perp \rightarrow P_\omega^{\bar{A}} x) \Rightarrow P_\omega^{A^*} x.$$

$$\text{lim } \alpha \quad : \quad P_\alpha^A x \Rightarrow \exists \beta < \alpha P_\beta^A x \stackrel{\text{ind hyp}}{\Rightarrow} \exists \beta < \alpha P_\omega^{A^*} x \Rightarrow P_\omega^{A^*} x.$$

For the successor case we note first that $P_\beta^{\bar{A}} t_j \rightarrow P_\omega^{\bar{A}} t_j$; this is seen as follows: for $\beta < \omega$ it follows by the fact that $\alpha < \beta \Rightarrow (P_\alpha^{\bar{A}} x \rightarrow P_\beta^{\bar{A}} x)$ (routine induction, using monotonicity of \bar{A}); for $\beta > \omega$ we recollect the fact that at stage ω the iteration of \bar{A} has reached its fixed-point.

$$\begin{aligned}
\alpha = \beta + 1 : P_{\beta+1}^A x &\Rightarrow A(P_\beta^A, x) \equiv \\
&\dots P_\beta^A s_i \dots J(\dots P_\beta^A t_j \dots) \quad \Rightarrow \text{ind hyp} \\
&\dots P_\omega^{A^*} s_i \dots J(\dots P_\beta^A t_j \dots) \quad \Rightarrow \text{increase} \\
&\dots P_\omega^{A^*} s_i \dots J(\dots J(P_\beta^A t_j) \dots) \quad \Rightarrow \text{lemma 4.1(ii)} \\
&\dots P_\omega^{A^*} s_i \dots J(\dots J(P_\beta^{\bar{A}} t_j) \dots) \quad \Rightarrow \text{lemma 3.4} \\
&\dots P_\omega^{A^*} s_i \dots J(J(\dots P_\beta^{\bar{A}} t_j \dots)) \quad \Rightarrow \text{idempotency} \\
&\dots P_\omega^{A^*} s_i \dots J(\dots P_\beta^{\bar{A}} t_j \dots) \quad \Rightarrow \text{since } P_\beta^{\bar{A}} t_j \rightarrow P_\omega^{\bar{A}} t_j \\
&\dots P_\omega^{A^*} s_i \dots J(\dots P_\omega^{\bar{A}} t_j \dots) \quad \Leftrightarrow \text{by definition} \\
A^*(P_\omega^{A^*}, x) &\Leftrightarrow P_{\omega+1}^{A^*} x \Leftrightarrow P_\omega^{A^*} x \text{ for } A^* \in \mathcal{P}[P, x].
\end{aligned}$$

(i)(\leftarrow): by induction on n we prove: $P_n^{A^*} x \rightarrow P_{\omega+n+1}^A$.

$$\begin{aligned}
n=0 : P_0^{A^*} x &\Leftrightarrow A^*(\lambda x \cdot \perp, x) \Leftrightarrow \\
&\dots (\lambda x \cdot \perp) s_i \dots J(\dots P_\omega^{\bar{A}} t_j \dots) \quad \Rightarrow \text{lemma 4.1(i)} \\
&\dots (\lambda x \cdot \perp) s_i \dots J(\dots P_\omega^A t_j \dots) \quad \Rightarrow \\
&\dots P_\omega^A s_i \dots J(\dots P_\omega^A t_j \dots) \quad \Leftrightarrow \text{by definition} \\
A(P_\omega^A, x) &\Leftrightarrow P_{\omega+1}^A.
\end{aligned}$$

$$\begin{aligned}
n+1 : P_{n+1}^{A^*} x &\Leftrightarrow A^*(P_n^{A^*}, x) \Leftrightarrow \\
&\dots P_n^{A^*} s_i \dots J(\dots P_\omega^{\bar{A}} t_j \dots) \quad \Rightarrow \text{ind hyp} \\
&\dots P_{\omega+n+1}^A s_i \dots J(\dots P_\omega^{\bar{A}} t_j \dots) \quad \Rightarrow \text{lemma 4.1(i)} \\
&\dots P_{\omega+n+1}^A s_i \dots J(\dots P_\omega^A t_j \dots) \quad \Rightarrow \text{monotonicity} \\
&\dots P_{\omega+n+1}^A s_i \dots J(\dots P_{\omega+n+1}^A t_j \dots) \quad \Leftrightarrow \text{by definition} \\
A(P_{\omega+n+1}^A, x) &\Leftrightarrow P_{\omega+n+2}^A.
\end{aligned}$$

Then $P_\omega^{A^*} x \Leftrightarrow \exists n P_n^{A^*} x \Rightarrow \exists n P_{\omega+n+1}^A x \Leftrightarrow P_{\omega+\omega}^A x \Rightarrow P_\infty^A x$.

(ii): see the preceding line.

□ (lemma 4.2)

Now theorem 3.2 follows:

– closure at $\omega + \omega$:

$$\begin{aligned}
A(P_{\omega+\omega}^A, x) &\Leftrightarrow P_{\omega+\omega+1}^A x \Rightarrow P_\infty^A x \Rightarrow \text{lemma 4.2(i)} \\
P_\infty^{A^*} x &\stackrel{\text{lemma 4.2(ii)}}{\Rightarrow} P_{\omega+\omega}^A x.
\end{aligned}$$

– definability:

$$P_\infty^A x \Leftrightarrow P_{\omega+\omega}^A x \Leftrightarrow P_\omega^{A^*} x \text{ and } P_\omega^{A^*} \text{ is HA-definable.}$$

□ (theorem 3.2)

5. EXTENSIONS

One of the limitations of our theorem is, that there figures at most one J -operator in an e.i.d. . When we try to admit more, and proceed by repeatedly treating the J -operators in the same way as we did our single J -operators, we encounter the following difficulty: one J -operator need to be shifted outward over another, while it is not generally true that $J_1(J_2(Q)) \rightarrow J_2(J_1(Q))$. Define

$$J_2 \leq J_1 : \Leftrightarrow J_1(J_2(Q)) \rightarrow J_2(J_1(Q)) \text{ read } J_2 \text{ precedes } J_1.$$

THEOREM 5.1. *For $A(P, x)$ containing two J -operators J_1 and J_2 , where $J_1 \leq J_2$ or $J_2 \leq J_1$, the following holds:*

$$P^A = P_{\omega+\omega+\omega+\omega}^A \text{ is HA-définable.}$$

PROOF. Define

$$\bar{A} : \equiv A \text{ where every } J_2 \text{ with } P \text{ in its scope has been deleted;}$$

$$A^* : \equiv A \text{ where every occurrence of } P \text{ in the scope of } J_2 \text{ has been replaced by } P_{\omega+\omega}^{\bar{A}}.$$

Then proceed in the same way as before.

□

I conclude with some examples of J -operators and a few easy relationships between them. The following are all J -operators:

$$I = \lambda Q \cdot Q$$

$$N = \lambda Q \cdot \neg \neg Q$$

$$D_R = \lambda Q \cdot Q \vee R$$

$$H_R = \lambda Q \cdot R \rightarrow Q$$

$$N_R = \lambda Q \cdot (Q \rightarrow R) \rightarrow R$$

$$N_R^{J_1} = \lambda Q \cdot N_R(J_1(Q))$$

$$M_R^{J_1 J_2} = \lambda Q \cdot (J_1(Q) \rightarrow R) \rightarrow J_2(Q) \text{ where } J_2(Q) \rightarrow J_1(Q) \text{ for all } Q.$$

It is not hard to establish that

$$N \leq J, I \leq J, H_{R_1} \leq H_{R_2}, D_{R_1} \leq D_{R_2}.$$

FACT 5.2.

$$J_1 \leq J_2 \Leftrightarrow J_1 \circ J_2 \text{ is a } J\text{-operator.}$$

PROOF.

(only if) straightforward; the condition $J_1 \leq J_2$ is used to get idempotency for $J_1 \circ J_2$.

(if) $J_2 J_1 Q \Rightarrow$ increase, monotonicity

$J_2 J_1 (J_2 Q) \Rightarrow$ increase

$J_1 (J_2 J_1 (J_2 Q)) \equiv (J_1 \circ J_2)(J_1 \circ J_2) Q = (J_1 \circ J_2) Q$ by the idempotency of $(J_1 \circ J_2)$.

□

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