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Elementary inductive definitions in HA: from strictly positive towards monotone

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ABSTRACT

A study of elementary inductive definitions (e.i.d.) in **HA**. Strictly positive e.i.d. have closure ordinals $\leq \omega$, and define predicates that are already definable in **HA**. We enlarge this class by adding so-called J-operators, for example $\neg\neg$. E.i.d. in this larger class have closure ordinals up to $\omega+\omega$, but they are conservative over **HA** w.r.t. definability.

1. INTRODUCTION

We shall consider as inductive definitions formulae in the language of **HA** expanded with a single one place predicate variable P, containing at most one numerical variable free. The meaning of such an inductive definition $A(P,x)$ is the least fixed-point of $A(P,x)$, i.e. a predicate P^A satisfying

(i): $\forall x(A(P^A,x) \leftrightarrow P^A x)$

(ii): $\forall x(A(Q,x) \rightarrow Qx) \rightarrow \forall x(P^A x \rightarrow Qx)$.

So the inductive definition specifies the closure conditions of the predicate it defines. The question is: for which $A(P,x)$ can we justify the existence of such a P^A? If $A(P,x)$ is *monotone*, i.e.

$$\forall x(Qx \rightarrow Rx) \rightarrow \forall x(A(Q,x) \rightarrow A(R,x)),$$

then we can approximate P^A from below; define

$$P_0^A x \quad :\Leftrightarrow A(\lambda x\cdot \bot, x)$$

$$P_{\beta+1}^A x :\Leftrightarrow A(P_\beta^A, x)$$

$$P^A_\lambda x \quad :\Leftrightarrow \exists\mu<\lambda P^A_\mu x, \text{ lim } \lambda$$

$$P^A_\infty x \quad :\Leftrightarrow \exists\mu P^A_\mu x$$

Note that for monotone $A(P,x)$ (i)$\leftarrow$is redundant: we have $A(P^A,x)\rightarrow P^A x$ by (i)$\rightarrow$, then by monotonicity we get $A(A(P^A,\cdot),x)\rightarrow A(P^A,x)$, and finally by (ii) $P^A x\rightarrow A(P^A,x)$.

Classically P^A exists and is equal to the least fixed-point of $A(P,x)$. An *elementary* inductive definition (e.i.d.) is an inductive definition without an unbounded universal quantifier occurring in front of a positive subformula containing P, and without an unbounded existential quantifier in front of a negative subformula containing P; the inductive definition must be monotone. Classically we know that for e.i.d. the approximation closes up at or before stage ω, so $P^A_\infty = P^A_\omega$. Intuitionistically, this is only true (in general) for strictly positive inductive definitions, i.e. formulae $A(P,x)$ built up from atomic formulae Pt, from **HA**-formulae φ (these do not contain P), by means of $\exists$, $\forall y<s$, $\wedge$, $\vee$.

Now we want to solve the following problems

(i): give neat ordinal bounds for arbitrary e.i.d., not only for the strictly positive ones

(ii): prove or refute: e.i.d. enhance the expressive power of **HA**.

I have no complete answer to these questions. I will describe special extensions of the class of strictly positive e.i.d., which do not enhance the expressive power of **HA**, while those e.i.d. may have a closure ordinal up to $\omega+\omega$. Those extensions are made by closing the strictly positive formulae under new operations, like $\neg\neg$. When we allow arbitrary monotone formulae, these problems look rather intractable. In particular, implication (with negative antecedent and positive consequent) seems rather tough.

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This article is a partial answer to a question, posed by Kreisel in [Kre63, p. 3.25]. I am indebted to prof. A.S. Troelstra for remembering it, and pointing it out to me.

CONVENTION

Throughout this article the symbols $\leftrightarrow$ resp. $\rightarrow$ and $\Leftrightarrow$ resp. $\Rightarrow$ stand for *provable* equivalence resp. consequence in a formal system. But only $\leftrightarrow$ and $\rightarrow$ are used as connectives in a formal language, while $\Leftrightarrow$ and $\Rightarrow$ denote equivalence resp. consequence relations between formulae.

2. EXAMPLES

2.1. CLOSURE AT $\omega+1$

An e.i.d. that closes up at stage $\omega+1$ (exactly). Let C be a nonrecursive RE-set, say

$$x\in C\leftrightarrow\exists z Texz;\text{ assume } Texz\rightarrow x\leq z.$$

Define, assuming that pairing is surjective:

$$A(P, \langle x, z \rangle) : \Leftrightarrow \exists m \leq z Texm \vee P\langle x, z+1 \rangle.$$

Then

$$P_0^A \langle x, z \rangle \Leftrightarrow \exists m \leq z Texm$$

$$P_1^A \langle x, z \rangle \Leftrightarrow \exists m \leq z Texm \vee P_0^A \langle x, z+1 \rangle \Leftrightarrow \exists m \leq z+1 Texm$$

$$\vdots$$

$$P_k^A \langle x, z \rangle \Leftrightarrow \exists m \leq z+k Texm$$

$$\vdots$$

$$P_\omega^A \langle x, z \rangle \Leftrightarrow \exists m Texm \Leftrightarrow x \in C.$$

We see quickly that $P_\omega^A = P_{\omega+1}^A$ and $P_k^A \neq P_\omega^A$. The last inequality follows from the fact that C is infinite and $Texz \rightarrow x \leq z$. Now we define, following [Kre63, pp. 3.6 and 3.24]:

$$B(P, x) : \Leftrightarrow A(P, x) \vee \neg\neg Px.$$

Then, for all $n < \omega$, $P_n^B x \leftrightarrow P_n^A x$, and P_n^A is recursive.

PROOF.

$$P_0^B x \quad \Leftrightarrow P_0^A x \vee \neg\neg \perp \Leftrightarrow P_0^A x \text{ and clearly } P_0^A \text{ is recursive.}$$

$$P_{n+1}^B x \Leftrightarrow A(P_n^B, x) \vee \neg\neg P_n^B x \Leftrightarrow \text{ind hyp}$$

$$A(P_n^A, x) \vee \neg\neg P_n^A x \Leftrightarrow \text{def, ind hyp}$$

$$P_{n+1}^A x \vee P_n^A x \qquad \Leftrightarrow P_{n+1}^A x, \text{ and } P_{n+1}^A \text{ is recursive.}$$

□

Consider now P_ω^B, $P_{\omega+1}^B$ and $P_{\omega+2}^B$:

$$P_\omega^B x \quad \Leftrightarrow \exists n P_n^B x \Leftrightarrow \exists n P_n^A x \Leftrightarrow P_\omega^A x.$$

$$P_{\omega+1}^B x \Leftrightarrow B(P_\omega^B, x) \Leftrightarrow A(P_\omega^A, x) \vee \neg\neg P_\omega^A x$$

$$\Leftrightarrow P_\omega^A x \vee \neg\neg P_\omega^A x \Leftrightarrow \neg\neg P_\omega^A x \not\Leftrightarrow P_\omega^A x, \text{ for } P_\omega^A \text{ is nonrecursive.}$$

$$P_{\omega+2}^B x \Leftrightarrow A(P_{\omega+1}^B, x) \vee \neg\neg P_{\omega+1}^B x \Leftrightarrow A(\neg\neg P_\omega^A, x) \Leftrightarrow \neg\neg\neg\neg P_\omega^A x$$

$$\Leftrightarrow \neg\neg P_\omega^A x \text{ because } A(\neg\neg P_\omega^A, x) \Rightarrow \neg\neg A(P_\omega^A, x) \Rightarrow \neg\neg P_\omega^A x.$$

It is possible to construe e.i.d. $C(P, x)$ that close up at stage $\omega + \omega$, by exploiting this trick.

□ (first example)

2.2. CLOSURE AT $\omega+\omega$

We give an e.i.d. with closure ordinal $\omega+\omega$. Let $\langle \dots \rangle$ be a coding of sequences of natural numbers. Let $A(P,x)$ be an e.i.d. that defines a nonrecursive $P^A = P^A_\omega$, while the P^A_k are recursive (cf. the first example); in addition, let $P^A \subseteq \{\langle x\rangle \mid x \in \mathbb{N}\}$, and let $A(P,x)$ be insensitive to numbers outside this set, i.e.

$$A(P,x) \leftrightarrow A(\lambda y \cdot Py \wedge \exists z(\langle z\rangle = y), x).$$

Define

$$B(P,x) :\equiv (A(P,x) \wedge \mathrm{lh}\, x = 1) \vee$$
$$\vee\, \exists y \exists z (Py \wedge \neg\neg A(P,z) \wedge \mathrm{lh}\, z = 1 \wedge x = y * z).$$

Then $P^B = P^B_{\omega+\omega}$, by the following lemmas, whose proofs are not particularly interesting and not too difficult. Sometimes I use set-theoretic notation like $x \in P^A_\omega$ for $P^A_\omega x$.

LEMMA 2.1. $P^B_\omega = \{\langle x_1, \dots, x_k\rangle \mid k \in \mathbb{N}, \langle x_i\rangle \in P^A_\omega, i = 1, \dots, k\}$.

LEMMA 2.2.

$$\begin{aligned} P^B_{\omega+n} = \{\langle x_1, \dots, x_k\rangle \mid & k > 0 \wedge \langle x_1\rangle \in P^A_\omega \\ & \wedge \forall i \in \{1, \dots, k \dot- n\} \langle x_i\rangle \in P^A_\omega \\ & \wedge \forall i \in \{k \dot- (n+1), \dots, k\} \langle x_i\rangle \in \neg\neg P^A_\omega\}. \end{aligned}$$

LEMMA 2.3. $x \in P^B_{\omega+n+1} \not\Leftrightarrow x \in P^B_{\omega+n}$.

LEMMA 2.4.

$$P^B_{\omega+\omega} = \bigcup_{n \in \omega} P^B_{\omega+n} =$$
$$= \{\langle x_1, \dots, x_k\rangle \mid k > 0 \wedge \langle x_1\rangle \in P^A_\omega \wedge \langle x_2\rangle, \dots, \langle x_k\rangle \in \neg\neg P^A_\omega\}.$$

It is clear from this construction, that the closure ordinal of B cannot be proved to be less than $\omega+\omega$.

3. J-OPERATORS

The following definition is meant as a generalization of the $\neg\neg$-operator (cf. [FS73, pp. 324–334]):

DEFINITION 3.1. *A* J-operator *is an operator* $J(\cdot)$, *on* **HA**-*formulae, that is* **HA**-*definable, and that satisfies*:

(i): $Q \to J(Q)$ (*increasing*)
(ii): $J(Q \wedge R) \leftrightarrow J(Q) \wedge J(R)$ ($\wedge$-*distributive*)
(iii): $J(J(Q)) \to J(Q)$ (*idempotent*)

Note that from (ii)($\to$) *follows*:

(iv): $(Q \to R) \to (J(Q) \to J(R))$ (*monotone*).

We do not allow J to have free variables.

DEFINITION 3.2. *$P[P]$ is the class of strictly positive formulae, i.e.:*
- *Pt, t a term, is a formula of $P[P]$*
- *a formula φ of the language of* **HA** *is a formula of $P[P]$*
- *$P[P]$ is closed under $\exists$, $\forall^{<}$, $\wedge$, $\vee$.*

$P(J)[P]$, J a J-operator, is defined analogously, except that $P(J)[P]$ is also closed under J.

FACT 3.1. *For $A(P,x) \in P[P,x]$, $P^A = P^A_\omega$ is* **HA***-definable.* See [TvD88, Vol I, pp. 145–152].

THEOREM 3.2. *For $A(P,x) \in P(J)[P,x]$, $P^A = P^A_{\omega+\omega}$ is* **HA***-definable.*

Before giving the proof, I will supply some technical lemmas and hint at the idea behind the proof.

LEMMA 3.3. (Shifting J to the outside)
(i): $J(P) \vee J(Q) \to J(P \vee Q)$
(ii): $J(P) \wedge J(Q) \to J(P \wedge Q)$
(iii): $\exists x J(A(x)) \to J(\exists x A(x))$
(iv): $\forall x < t J(A(x)) \to J(\forall x < t A(x))$.

PROOF
(i): $\left.\begin{matrix} P \to P \vee Q \\ Q \to P \vee Q \end{matrix}\right\} \overset{\text{monotonicity}}{\Rightarrow} \left.\begin{matrix} J(P) \to J(P \vee Q) \\ J(Q) \to J(P \vee Q) \end{matrix}\right\} \Rightarrow J(P) \vee J(Q) \to J(P \vee Q)$
(ii): by $\wedge$-distributivity($\leftarrow$)
(iii): $A(x) \to \exists x A(x) \Rightarrow J(A(x)) \to J(\exists x A(x)) \Rightarrow \exists x J(A(x)) \to J(\exists x A(x))$
(iv): let J-SHIFT(y) denote the following schema:

$$\forall x(x < y \to J(A(x))) \to J(\forall x(x < y \to A(x))),\ y \notin FV(A).$$

We prove $\forall y$ J-SHIFT(y) by induction:

$\forall x(x < 0 \to A(x))$, so by increase: $J(\forall x(x < 0 \to A(x)))$.

$\forall x(x < Sy \to J(A(x))$	$\Rightarrow$ "**HA**"
$\forall x(x < y \to J(A(x))) \wedge J(A(y))$	$\Rightarrow$ ind hyp
$J(\forall x(x < y \to A(x))) \wedge J(A(y))$	$\Rightarrow$ $\wedge$-distributivity
$J(\forall x(x < y \to A(x)) \wedge J(A(y)))$	$\Rightarrow$ "**HA** under J"
$J(\forall x(x < Sy \to A(x)))$.	

We conclude: for any term t:

$$\forall x(x < t \to J(A(x))) \to J(\forall x(x < t \to A(x))).$$

□ (lemma 3.3)

The comment "**HA**" means: by reasoning in **HA**; "**HA** under J" means: by reasoning in **HA** in the scope of J; this is justified by the fact that J is increasing and monotone.

DEFINITION 3.3. *Let $A(P)$ be a $P(J)[P]$-formula. Occurrences of subformulae, used in the construction of $A(P)$,* according to the definition of $P(J)[P]$, *are called* components.

Remark that a $P(J)[P]$-formula is monotone in its components, because $\exists$, $\forall^{<}$, $\wedge$, $\vee$, J are all monotone connectives.

LEMMA 3.4. *Let $A(P)$ be a $P(J)[P]$-formula. Let C be a component of $A(P)$ of the form $J(B(P))$, with at least one occurrence of P. Let $A'(P)$ be obtained from A by replacing that component $J(B(P))$ by $B(P)$. Then $A(P) \rightarrow J(A'(P))$. I.e.*

$$A(P) \equiv \ldots J(B(P)) \ldots$$

$$J(A'(P)) \equiv J(\ldots B(P) \ldots).$$

PROOF. Easy, by induction on the structure of $A(P)$. In fact, this is nothing else than repeatedly shifting J outwards, using the fact that a component occurs only in scopes of $\wedge$, $\vee$, $\exists$, $\forall^{<}$, J, and applying lemma 3.3.

□

4. DECOMPOSITION OF THE APPROXIMATION PROCESS

DEFINITION 4.1. *Let $A(P, x)$ be a $P(J)[P]$-formula.*

$\bar{A}$ $:\equiv A$ *where every J with P in its scope has been deleted*;

A^* $:\equiv A$ *where every occurrence of P in the scope of J has been replaced by $P^{\bar{A}}_{\omega}$; so:*

$$A(P) \equiv \ldots Ps_i \ldots J(\ldots Pt_j \ldots)$$

$$\bar{A}(P) \equiv \ldots Ps_i \ldots \quad \ldots Pt_j \ldots$$

$$A^*(P) \equiv \ldots Ps_i \ldots J(\ldots P^{\bar{A}}_{\omega} t_j \ldots).$$

REMARK

$\bar{A} \in P[P, x]$, so $P^{\bar{A}} = P^{\bar{A}}_{\omega}$ is **HA**-definable by the fact above; it follows that A^* is a $P[P, x]$-formula, so $P^{A^*} = P^{A^*}_{\omega}$ is **HA**-definable too.

The idea of the proof is emerging: instead of iterating $A(P, x)$ indefinitely, we split the process in iterations that continue at most till stage ω. In the first iteration we neglect the J-operator completely, then we administer its effect one time; the second iteration also goes on without J-operator. The reason that this suffices, is mainly the idempotency of the J-operator.

LEMMA 4.1. *Let* $A(P, x) \in P(J)[P, x]$. *Then*
(i): $P_\alpha^{\bar{A}}x \to P_\alpha^A x$
(ii): $J(P_\alpha^A x) \to J(P_\alpha^{\bar{A}} x)$.

PROOF. (i) follows from $\bar{A} \to A$, (ii) from $J(A) \to J(\bar{A})$, both by induction on α. Ad (i): A is obtained from $\bar{A}$ by replacing components B by $J(B)$. Use increase ($B \to J(B)$) and monotonicity in components. Ad (ii): this is seen as follows: by repeatedly applying lemma 3.4 we have $A \to J(\bar{A})$; then, by monotonicity $J(A) \to J(J(\bar{A}))$ and by idempotency $J(A) \to J(\bar{A})$. Let us now carry out the induction for (ii):

$\alpha = 0$: $J(P_0^A x) \overset{\text{by def}}{\equiv} J(A(\lambda x \cdot \bot, x))$ $\Rightarrow$ for $J(A) \to J(\bar{A})$, see above
$J(\bar{A}(\lambda x \cdot \bot, x)) \overset{\text{by def}}{\equiv} J(P_0^{\bar{A}}, x)$.

$\alpha = \beta + 1$: $J(P_{\beta+1}^A x) \overset{\text{by def}}{\equiv} J(A(P_\beta^A, x))$ $\Rightarrow$ for $J(A) \to J(\bar{A})$, see above
$J(\bar{A}(P_\beta^A, x))$ $\Rightarrow$ $\bar{A}$ monotone, J increasing
$J(\bar{A}(J(P_\beta^A), x))$ $\Rightarrow$ ind hyp
$J(\bar{A}(J(P_\beta^{\bar{A}}), x))$ $\Rightarrow$ lemma 3.4
$J(J(\bar{A}(P_\beta^{\bar{A}}, x))$ $\Rightarrow$ idempotency
$J(P_{\beta+1}^{\bar{A}} x)$.

lim α : $J(P_\alpha^A x) \overset{\text{by def}}{\equiv} J(\exists\beta < \alpha P_\beta^A x)$ $\Rightarrow$ J increasing
$J(\exists\beta < \alpha J(P_\beta^A x))$ $\Rightarrow$ ind hyp, monotonicity of J
$J(\exists\beta < \alpha J(P_\beta^{\bar{A}} x))$ $\Rightarrow$ lemma 3.4
$J(J(\exists\beta < \alpha P_\beta^{\bar{A}} x))$ $\Rightarrow$ idempotency
$J(P_\alpha^{\bar{A}} x)$.

□

LEMMA 4.2. *Let* $A(P, x) \in P(J)[P, x]$. *Then*
(i): $P_\infty^A x \leftrightarrow P_\omega^{A^*} x$
(ii): $P_\omega^{A^*} x \leftrightarrow P_{\omega+\omega}^A x$.

PROOF. (i)($\to$): by induction on α we prove $P_\alpha^A x \to P_\omega^{A^*} x$.

$\alpha = 0$: $P_0^A x \Leftrightarrow A(\lambda x \cdot \bot, x) \Rightarrow P_0^{A^*} x$ (since $\bot \to P_\omega^{\bar{A}} x) \Rightarrow P_\omega^{A^*} x$.

lim α : $P_\alpha^A x \Leftrightarrow \exists\beta < \alpha P_\beta^A x \overset{\text{ind hyp}}{\Rightarrow} \exists\beta < \alpha P_\omega^{A^*} x \Rightarrow P_\omega^{A^*} x$.

For the successor case we note first that $P_\beta^{\bar{A}} t_j \to P_\omega^{\bar{A}} t_j$; this is seen as follows: for $\beta < \omega$ it follows by the fact that $\alpha < \beta \Rightarrow (P_\alpha^{\bar{A}} x \to P_\beta^{\bar{A}} x)$ (routine induction, using monotonicity of $\bar{A}$); for $\beta > \omega$ we recollect the fact that at stage ω the iteration of $\bar{A}$ has reached its fixed-point.

$\alpha=\beta+1$: $P^A_{\beta+1}x \Rightarrow A(P^A_\beta, x) \equiv$

$\dots P^A_\beta s_i \dots J(\dots P^A_\beta t_j \dots)$ $\Rightarrow$ ind hyp

$\dots P^{A*}_\omega s_i \dots J(\dots P^A_\beta t_j \dots)$ $\Rightarrow$ increase

$\dots P^{A*}_\omega s_i \dots J(\dots J(P^A_\beta t_j)\dots)$ $\Rightarrow$ lemma 4.1(ii)

$\dots P^{A*}_\omega s_i \dots J(\dots J(P^{\bar{A}}_\beta t_j)\dots)$ $\Rightarrow$ lemma 3.4

$\dots P^{A*}_\omega s_i \dots J(J(\dots P^{\bar{A}}_\beta t_j \dots))$ $\Rightarrow$ idempotency

$\dots P^{A*}_\omega s_i \dots J(\dots P^{\bar{A}}_\beta t_j \dots)$ $\Rightarrow$ since $P^{\bar{A}}_\beta t_j \rightarrow P^{\bar{A}}_\omega t_j$

$\dots P^{A*}_\omega s_i \dots J(\dots P^{\bar{A}}_\omega t_j \dots)$ $\Leftrightarrow$ by definition

$A^*(P^{A*}_\omega, x) \Leftrightarrow P^{A*}_{\omega+1}x \Leftrightarrow P^{A*}_\omega x$ for $A^* \in \mathscr{P}[P, x]$.

(i)($\leftarrow$): by induction on n we prove: $P^{A*}_n x \rightarrow P^A_{\omega+n+1}x$.

$n=0$: $P^{A*}_0 x \Leftrightarrow A^*(\lambda x \cdot \bot, x) \Leftrightarrow$

$\dots(\lambda x\cdot \bot)s_i \dots J(\dots P^{\bar{A}}_\omega t_j \dots)$ $\Rightarrow$ lemma 4.1(i)

$\dots(\lambda x\cdot \bot)s_i \dots J(\dots P^A_\omega t_j \dots)$ $\Rightarrow$

$\dots P^A_\omega s_i \dots J(\dots P^A_\omega t_j \dots)$ $\Leftrightarrow$ by definition

$A(P^A_\omega, x) \Leftrightarrow P^A_{\omega+1}x$.

$n+1$: $P^{A*}_{n+1}x \Leftrightarrow A^*(P^{A*}_n, x) \Leftrightarrow$

$\dots P^{A*}_n s_i \dots J(\dots P^{\bar{A}}_\omega t_j \dots)$ $\Rightarrow$ ind hyp

$\dots P^A_{\omega+n+1} s_i \dots J(\dots P^{\bar{A}}_\omega t_j \dots)$ $\Rightarrow$ lemma 4.1(i)

$\dots P^A_{\omega+n+1} s_i \dots J(\dots P^A_\omega t_j \dots)$ $\Rightarrow$ monotonicity

$\dots P^A_{\omega+n+1} s_i \dots J(\dots P^A_{\omega+n+1} t_j \dots)$ $\Leftrightarrow$ by definition

$A(P^A_{\omega+n+1}, x) \Leftrightarrow P^A_{\omega+n+2}x$.

Then $P^{A*}_\omega x \Leftrightarrow \exists n P^{A*}_n x \Rightarrow \exists n P^A_{\omega+n+1}x \Leftrightarrow P^A_{\omega+\omega}x \Rightarrow P^A_\infty x$.
(ii): see the preceeding line.

□ (lemma 4.2)

Now theorem 3.2 follows:
- closure at $\omega+\omega$:

$$A(P^A_{\omega+\omega}, x) \Leftrightarrow P^A_{\omega+\omega+1}x \Rightarrow P^A_\infty x \Rightarrow \text{lemma 4.2(i)}$$
$$P^{A*}_\omega x \overset{\text{lemma 4.2(ii)}}{\Rightarrow} P^A_{\omega+\omega}x.$$

- definability:

$$P^A_\infty x \Leftrightarrow P^A_{\omega+\omega}x \Leftrightarrow P^{A*}_\omega x \text{ and } P^{A*}_\omega \text{ is } \mathbf{HA}\text{-definable.}$$

□ (theorem 3.2)

5. EXTENSIONS

One of the limitations of our theorem is, that there figures at most one J-operator in an e.i.d.. When we try to admit more, and proceed by repeatedly treating the J-operators in the same way as we did our single J-operators, we encounter the following difficulty: one J-operator need to be shifted outward over another, while it is not generally true that $J_1(J_2(Q)) \rightarrow J_2(J_1(Q))$. Define

$$J_2 \leq J_1 : \Leftrightarrow J_1(J_2(Q)) \rightarrow J_2(J_1(Q)) \text{ read } J_2 \text{ preceeds } J_1.$$

THEOREM 5.1. *For $A(P, x)$ containing two J-operators J_1 and J_2, where $J_1 \leq J_2$ or $J_2 \leq J_1$, the following holds:*

$$P^A = P^A_{\omega+\omega+\omega+\omega} \text{ is } \mathbf{HA}\text{-definable.}$$

PROOF. Define

$\bar{A}$: $\equiv A$ where every J_2 with P in its scope has been deleted;

A^*: $\equiv A$ where every occurrence of P in the scope of J_2 has been replaced by $P^{\bar{A}}_{\omega+\omega}$.

Then proceed in the same way as before.

□

I conclude with some examples of J-operators and a few easy relationships between them. The following are all J-operators:

$$I \quad = \lambda Q \cdot Q$$

$$N \quad = \lambda Q \cdot \neg\neg Q$$

$$D_R \quad = \lambda Q \cdot Q \vee R$$

$$H_R \quad = \lambda Q \cdot R \rightarrow Q$$

$$N_R \quad = \lambda Q \cdot (Q \rightarrow R) \rightarrow R$$

$$N_R^{J_1} \quad = \lambda Q \cdot N_R(J_1(Q))$$

$$M_R^{J_1 J_2} = \lambda Q \cdot (J_1(Q) \rightarrow R) \rightarrow J_2(Q) \text{ where } J_2(Q) \rightarrow J_1(Q) \text{ for all } Q.$$

It is not hard to establish that

$$N \leq J,\ I \leq J,\ H_{R_1} \leq H_{R_2},\ D_{R_1} \leq D_{R_2}.$$

FACT 5.2.

$$J_1 \leq J_2 \Leftrightarrow J_1 \circ J_2 \text{ is a } J\text{-operator.}$$

PROOF.

(only if) straightforward; the condition $J_1 \leq J_2$ is used to get idempotency for $J_1 \circ J_2$.

(if) $J_2 J_1 Q \quad \Rightarrow$ increase, monotonicity

$J_2 J_1 (J_2 Q) \quad \Rightarrow$ increase

$J_1(J_2 J_1 (J_2 Q)) \equiv (J_1 \circ J_2)(J_1 \circ J_2) Q \Rightarrow (J_1 \circ J_2) Q$ by the idempotency of $(J_1 \circ J_2)$.

□

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